

A Kastler-Kalau-Walze Type Theorem for 5-dimensional Manifolds with Boundary

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Abstract

The Kastler-Kalau-Walze theorem, announced by Alain Connes, shows that the Wodzicki residue of the inverse square of the Dirac operator is proportional to the Einstein-Hilbert action of general relativity. In this paper, we prove a Kastler-Kalau-Walze type theorem for 5-dimensional manifolds with boundary.

Keywords: Dirac operators; Noncommutative residue for manifolds with boundary.

1. Introduction

The noncommutative residue found in [1, 2] plays a prominent role in noncommutative geometry. For one-dimensional manifolds, the noncommutative residue was discovered by Adler [3] in connection with geometric aspects of nonlinear partial differential equations. For arbitrary closed compact n -dimensional manifolds, the noncommutative residue was introduced by Wodzicki in [2] using the theory of zeta functions of elliptic pseudodifferential operators. In [4], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Furthermore, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action in [5]. Let s be the scalar curvature and Wres denote the noncommutative residue. Then the Kastler-Kalau-Walze theorem gives an operator-theoretic explanation of the gravitational action and says that for a 4-dimensional closed spin manifold, there exists a constant c_0 , such that

$$\text{Wres}(D^{-2}) = c_0 \int_M \text{sdvol}_M.$$

In [6], Kastler gave a brute-force proof of this theorem. In [7], Kalau and Walze proved this theorem in the normal coordinates system simultaneously. And then, Ackermann proved that the Wodzicki residue $\text{Wres}(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of D^2 in [8].

On the other hand, Fedosov etc. defined a noncommutative residue on Boutet de Monvel's algebra and proved that it was a unique continuous trace in [9]. In [10], Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. For an oriented spin manifold M with boundary ∂M , by the composition formula in Boutet de Monvel's algebra and the definition of $\widetilde{\text{Wres}}$ [11], $\widetilde{\text{Wres}}[(\pi^+ D^{-1})^2]$ should be the sum of two terms from interior and boundary of M , where $\pi^+ D^{-1}$ is an element in Boutet de Monvel's algebra [11]. It is well known that the gravitational action for manifolds with boundary is also the sum of two terms from interior and boundary of M [12]. Considering the Kastler-Kalau-Walze Theorem for manifolds without boundary, then the term from interior is proportional to gravitational action from interior, so it is natural to hope to get the gravitational action for manifolds with boundary

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by computing $\widetilde{\text{Wres}}[(\pi^+ D^{-1})^2]$. Based on the motivation, Wang [13] proved a Kastler-Kalau-Walze type theorem for 4-dimensional spin manifolds with boundary

$$\widetilde{\text{Wres}}[(\pi^+ D^{-1})^2] = -\frac{\Omega_3}{3} \int_M \text{sdvol}_M,$$

where Ω_3 is the canonical volume of S^3 . Furthermore, Wang [15] found a Kastler-Kalau-Walze type theorem for higher dimensional manifolds with boundary and generalized the definition of lower dimensional volumes in [14] to manifolds with boundary. For 5-dimensional spin manifolds with boundary [15], Wang get

$$\widetilde{\text{Wres}}[(\pi^+ D^{-2})^2] = \frac{\pi i}{2} \Omega_2 \text{vol}_{\partial M},$$

and for 6-dimensional spin manifolds with boundary

$$\widetilde{\text{Wres}}[(\pi^+ D^{-2})^2] = -\frac{5\Omega_5}{3} \int_M \text{sdvol}_M.$$

In order to get the boundary term, we computed the lower dimensional volume $\text{Vol}_6^{(1,3)}$ for 6-dimensional spin manifolds with boundary associated with D^{-1} , D^{-3} in [17], and obtained the volume with the boundary term

$$\widetilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}] = -\frac{5\Omega_4}{3} \int_M \text{sdvol}_M + \pi \Omega_3 \int_{\partial M} K d\text{vol}_{\partial M},$$

where K is the extrinsic curvature.

Recently, Ackermann and Tolksdorf [18] proved a generalized version of the well-known Lichnerowicz formula for the square of the most general Dirac operator with torsion D_T on an even-dimensional spin manifold associated to a metric connection with torsion. Meanwhile, Pfäffle and Stephan considered compact Riemannian spin manifolds without boundary equipped with orthogonal connections, and investigated the induced Dirac operators in [19]. In [20], Pfäffle and Stephan considered orthogonal connections with arbitrary torsion on compact Riemannian manifolds, and for the induced Dirac operators, twisted Dirac operators and Dirac operators of Chamseddine-Connes type they computed the spectral action. For the associated Dirac operators with torsion D_T^* , D_T [21], we got the Kastler-Kalau-Walze theorem associated to Dirac operators with torsion on 4-dimensional compact manifolds with boundary

$$\widetilde{\text{Wres}}[\pi^+ (D_T^*)^{-1} \circ \pi^+ D_T^{-1}] = -\frac{1}{48\pi^2} \int_M \tilde{R}(x) dx - \int_{\partial M} \sum_i A_{iin} \pi \Omega_2 dx',$$

where definitions of $\tilde{R}(x)$, A_{iin} , see [20]. In addition, we proved the Kastler-Kalau-Walze type theorems for foliations with or without boundary associated with sub-Dirac operators in [16]

$$\widetilde{\text{Wres}}[(\pi^+ D_F^{-1})^2] = -\frac{1}{24\sqrt{2} \cdot 2^p \pi^{p+\frac{q}{2}+1}} \int_M s_M d\text{vol}_M.$$

In fact, in previous papers, we computed $\widetilde{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}]$ for n -dimensional spin manifolds with boundary in case of $n - p_1 - p_2 \leq 2$. In the present paper, we shall restrict our attention to the case of $n - p_1 - p_2 = 3$. We compute $\widetilde{\text{Wres}}[(\pi^+ D^{-1})^2]$ for 5-dimensional manifolds with boundary. Our main result is as follows.

Main Theorem: The following identity holds

$$\widetilde{\text{Wres}}[(\pi^+ D^{-1})^2] = \frac{\pi^3}{16} \int_{\partial M} \left(\frac{225}{64} K^2 + \frac{29}{4} s_M|_{\partial M} + \left(\frac{197}{12} + 3i \right) s_{\partial M} \right) d\text{vol}_{\partial M}.$$

where s_M , $s_{\partial M}$ are respectively scalar curvatures on M and ∂M . Compared with the previous results, up to the extrinsic curvature, the scalar curvature on ∂M and the scalar curvature on M appear in the boundary

term. This case essentially makes the whole calculations more difficult, and the boundary term is the sum of fifteen terms. As in computations of the boundary term, we shall consider some new traces of multiplication of Clifford elements. And the inverse 3-order symbol of the Dirac operator and higher derivatives of -1-order, -2-order symbols of the Dirac operators will be extensively used.

This paper is organized as follows: In Section 2, we define lower dimensional volumes of compact Riemannian manifolds with boundary. In Section 3, for 5-dimensional spin manifolds with boundary and the associated Dirac operators, we compute $\widetilde{\text{Wres}}[(\pi^+ D^{-1})^2]$ and get a Kastler-Kalau-Walze type theorem in this case.

2. Lower-Dimensional Volumes of Spin Manifolds with boundary

In this section we consider an n -dimensional oriented Riemannian manifold (M, g^M) with boundary ∂M equipped with a fixed spin structure. We assume that the metric g^M on M has the following form near the boundary

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \quad (2.1)$$

where $g^{\partial M}$ is the metric on ∂M . Let $U \subset M$ be a collar neighborhood of ∂M which is diffeomorphic $\partial M \times [0, 1)$. By the definition of $h(x_n) \in C^\infty([0, 1))$ and $h(x_n) > 0$, there exists $\tilde{h} \in C^\infty((-\varepsilon, 1))$ such that $\tilde{h}|_{[0, 1)} = h$ and $\tilde{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric \hat{g} on $\hat{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0]$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$\hat{g} = \frac{1}{\tilde{h}(x_n)} g^{\partial M} + dx_n^2, \quad (2.2)$$

such that $\hat{g}|_M = g$. We fix a metric \hat{g} on the \hat{M} such that $\hat{g}|_M = g$.

Let us give the expression of Dirac operators near the boundary. Set $\tilde{E}_n = \frac{\partial}{\partial x_n}$, $\tilde{E}_j = \sqrt{h(x_n)} E_j$ ($1 \leq j \leq n-1$), where $\{E_1, \dots, E_{n-1}\}$ are orthonormal basis of $T\partial M$. Let ∇^L denote the Levi-civita connection about g^M . In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\tilde{E}_1, \dots, \tilde{E}_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(\tilde{E}_1, \dots, \tilde{E}_n)^t = (\omega_{s,t})(\tilde{E}_1, \dots, \tilde{E}_n)^t. \quad (2.3)$$

The Dirac operator is defined by

$$D = \sum_{j=1}^n c(\tilde{E}_j) \left[\tilde{E}_j + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{E}_j) c(\tilde{E}_s) c(\tilde{E}_t) \right]. \quad (2.4)$$

By Lemma 6.1 in [16] and Proposition 2.2, Proposition 2.4 in [22], we have

Lemma 2.1. *Let $f = \frac{1}{\sqrt{h}}$ and $\tilde{M} = I \times_f M$ be a Riemannian manifold with the metric $g_f = dx_n^2 + f^2(x_n)g$. For vector fields X, Y in $\mathcal{L}(M)$, then*

$$(1) \quad \tilde{\nabla}_{\partial_{x_n}} \partial_{x_n} = 0; \quad (2.5)$$

$$(2) \quad \tilde{\nabla}_{\partial_{x_n}} X = \tilde{\nabla}_X \partial_{x_n} = (\ln f)' X; \quad (2.6)$$

$$(3) \quad \nabla_X Y = \nabla_X^M Y - \frac{g(X, Y)}{f} \text{grad}(f). \quad (2.7)$$

Denote $A_{js}^t = 2\langle \nabla_{E_j}^{L, \partial M} E_s, E_t \rangle$, then we obtain

Lemma 2.2. *The following identity holds:*

$$(1) \langle \nabla_{\widetilde{E}_i}^L \partial_{x_n}, \widetilde{E}_j \rangle = -\frac{h'}{2h}; \quad (2.8)$$

$$(2) \langle \nabla_{\widetilde{E}_i}^L \widetilde{E}_j, \partial_{x_n} \rangle = \frac{h'}{2h}; \quad (2.9)$$

$$(3) \langle \nabla_{\widetilde{E}_j}^L \widetilde{E}_s, \widetilde{E}_t \rangle = \frac{\sqrt{h}}{2} A_{js}^t. \quad (2.10)$$

Others are zeros.

By Lemma 2.2, we have

Definition 2.3. *The following identity holds in the coordinates near the boundary*

$$D = \sum_{\beta=1}^n c(\widetilde{E}_\beta) \widetilde{E}_\beta - \frac{h'}{h} c(\mathbf{d}x_n) + \frac{\sqrt{h}}{8} \sum_{s,\alpha} A_{\beta s}^\alpha c(\widetilde{E}_\beta) c(\widetilde{E}_s) c(\widetilde{E}_\alpha). \quad (2.11)$$

To define the lower dimensional volume, some basic facts and formulae about Boutet de Monvel's calculus which can be found in Sec.2 in [11] are needed.

Let

$$F : L^2(\mathbf{R}_t) \rightarrow L^2(\mathbf{R}_v); F(u)(v) = \int e^{-ivt} u(t) dt \quad (2.12)$$

denote the Fourier transformation and $\Phi(\overline{\mathbf{R}^+}) = r^+ \Phi(\mathbf{R})$ (similarly define $\Phi(\overline{\mathbf{R}^-})$), where $\Phi(\mathbf{R})$ denotes the Schwartz space and

$$r^+ : C^\infty(\mathbf{R}) \rightarrow C^\infty(\overline{\mathbf{R}^+}); f \rightarrow f|_{\overline{\mathbf{R}^+}}; \overline{\mathbf{R}^+} = \{x \geq 0; x \in \mathbf{R}\}. \quad (2.13)$$

We define $H^+ = F(\Phi(\overline{\mathbf{R}^+}))$; $H_0^- = F(\Phi(\overline{\mathbf{R}^-}))$ which are orthogonal to each other. We have the following property: $h \in H^+$ (H_0^-) iff $h \in C^\infty(\mathbf{R})$ which has an analytic extension to the lower (upper) complex half-plane $\{\text{Im} \xi < 0\}$ ($\{\text{Im} \xi > 0\}$) such that for all nonnegative integer l ,

$$\frac{d^l h}{d\xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{d^l}{d\xi^l} \left(\frac{c_k}{\xi^k} \right) \quad (2.14)$$

as $|\xi| \rightarrow +\infty, \text{Im} \xi \leq 0$ ($\text{Im} \xi \geq 0$).

Let H' be the space of all polynomials and $H^- = H_0^- \oplus H'$; $H = H^+ \oplus H^-$. Denote by π^+ (π^-) respectively the projection on H^+ (H^-). For calculations, we take $H = \tilde{H} = \{\text{rational functions having no poles on the real axis}\}$ (\tilde{H} is a dense set in the topology of H). Then on \tilde{H} ,

$$\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \rightarrow 0^-} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi, \quad (2.15)$$

where Γ^+ is a Jordan close curve included $\text{Im} \xi > 0$ surrounding all the singularities of h in the upper half-plane and $\xi_0 \in \mathbf{R}$. Similarly, define π' on \tilde{H} ,

$$\pi' h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi) d\xi. \quad (2.16)$$

So, $\pi'(H^-) = 0$. For $h \in H \cap L^1(\mathbf{R})$, $\pi' h = \frac{1}{2\pi} \int_{\mathbf{R}} h(v) dv$ and for $h \in H^+ \cap L^1(\mathbf{R})$, $\pi' h = 0$.

Let M be an n -dimensional compact oriented manifold with boundary ∂M . Denote by \mathcal{B} Boutet de Monvel's algebra, we recall the main theorem in [9].

Theorem 2.4. (Fedosov-Golse-Leichtnam-Schrohe) *Let X and ∂X be connected, $\dim X = n \geq 3$, $A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in \mathcal{B}$, and denote by p, b and s the local symbols of P, G and S respectively. Define:*

$$\begin{aligned} \widetilde{\text{Wres}}(A) &= \int_X \int_{\mathbf{S}} \text{tr}_E [p_{-n}(x, \xi)] \sigma(\xi) dx \\ &\quad + 2\pi \int_{\partial X} \int_{\mathbf{S}'} \{ \text{tr}_E [(\text{tr} b_{-n})(x', \xi')] + \text{tr}_F [s_{1-n}(x', \xi')] \} \sigma(\xi') dx', \end{aligned} \quad (2.17)$$

Then a) $\widetilde{\text{Wres}}([A, B]) = 0$, for any $A, B \in \mathcal{B}$; b) It is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

Let p_1, p_2 be nonnegative integers and $p_1 + p_2 \leq n$. Then by Sec 2.1 of [13], we have

Definition 2.5. *Lower-dimensional volumes of spin manifolds with boundary are defined by*

$$\text{Vol}_n^{(p_1, p_2)} M := \widetilde{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}]. \quad (2.18)$$

Denote by $\sigma_l(A)$ the l -order symbol of an operator A . An application of (2.1.4) in [11] shows that

$$\widetilde{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}] = \int_M \int_{|\xi|=1} \text{trace}_{S(TM)} [\sigma_{-n}(D^{-p_1-p_2})] \sigma(\xi) d\mathbf{x} + \int_{\partial M} \Phi, \quad (2.19)$$

where

$$\begin{aligned} \Phi &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \text{trace}_{S(TM)} \left[\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+(D^{-p_1})(x', 0, \xi', \xi_n) \right. \\ &\quad \left. \times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(D^{-p_2})(x', 0, \xi', \xi_n) \right] d\xi_n \sigma(\xi') d\mathbf{x}', \end{aligned} \quad (2.20)$$

and the sum is taken over $r - k + |\alpha| + \ell - j - 1 = -n, r \leq -p_1, \ell \leq -p_2$.

3. A Kastler-Kalau-Walze type theorem for 5-dimensional spin manifolds with boundary

In this section, we compute the lower dimensional volume for 5-dimensional compact manifolds with boundary and get a Kastler-Kalau-Walze type formula in this case. From now on we always assume that M carries a spin structure so that the spinor bundle and the Dirac operator are defined on M .

The following proposition is the key of the computation of lower-dimensional volumes of spin manifolds with boundary.

Proposition 3.1. [15] *The following identity holds:*

$$1) \text{ When } p_1 + p_2 = n, \text{ then, } \text{Vol}_n^{(p_1, p_2)} M = c_0 \text{Vol}_M; \quad (3.1)$$

$$2) \text{ when } p_1 + p_2 \equiv n \pmod{1}, \text{ Vol}_n^{(p_1, p_2)} M = \int_{\partial M} \Phi. \quad (3.2)$$

Nextly, for 5-dimensional spin manifolds with boundary, we compute $\text{Vol}_5^{(1,1)}$. By Proposition 3.1, we have

$$\widetilde{\text{Wres}}[(\pi^+ D^{-1})^2] = \int_{\partial M} \Phi. \quad (3.3)$$

Recall the Dirac operator D of the definition 2.3. Write

$$D_x^\alpha = (-\sqrt{-1})^{|\alpha|} \partial_x^\alpha; \quad \sigma(D) = p_1 + p_0; \quad \sigma(D^{-1}) = \sum_{j=1}^{\infty} q_{-j}. \quad (3.4)$$

By the composition formula of pseudodifferential operators, then we have

$$\begin{aligned}
1 = \sigma(D \circ D^{-1}) &= \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} [\sigma(D)] D_x^{\alpha} [\sigma(D^{-1})] \\
&= (p_1 + p_0)(q_{-1} + q_{-2} + q_{-3} + \cdots) \\
&\quad + \sum_j (\partial_{\xi_j} p_1 + \partial_{\xi_j} p_0)(D_{x_j} q_{-1} + D_{x_j} q_{-2} + D_{x_j} q_{-3} + \cdots) \\
&= p_1 q_{-1} + (p_1 q_{-2} + p_0 q_{-1} + \sum_j \partial_{\xi_j} p_1 D_{x_j} q_{-1}) \\
&\quad + (p_1 q_{-3} + p_0 q_{-2} + \sum_j \partial_{\xi_j} p_1 D_{x_j} q_{-2}) + \cdots.
\end{aligned} \tag{3.5}$$

Thus, we get

$$q_{-1} = p_1^{-1}; \tag{3.6}$$

$$q_{-2} = -p_1^{-1} \left[p_0 p_1^{-1} + \sum_j \partial_{\xi_j} p_1 D_{x_j} (p_1^{-1}) \right]; \tag{3.7}$$

$$q_{-3} = -p_1^{-1} \left[p_0 q_{-2} + \sum_{j=1}^{n-1} c(\mathbf{d}x_j) \partial_{x_j} q_{-2} + c(\mathbf{d}x_n) \partial_{x_n} q_{-2} \right]. \tag{3.8}$$

By Lemma 2.1 in [13], we have

Lemma 3.2. *The symbol of the Dirac operator*

$$\sigma_{-1}(D^{-1}) = \frac{\sqrt{-1}c(\xi)}{|\xi|^2}; \tag{3.9}$$

$$\sigma_{-2}(D^{-1}) = \frac{c(\xi)p_0 c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(\mathbf{d}x_j) \left[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi) \partial_{x_j}(|\xi|^2) \right], \tag{3.10}$$

where

$$p_0 = -\frac{h'}{h} c(\mathbf{d}x_n) + \frac{\sqrt{h}}{8} \sum_{s,\alpha} A_{\beta s}^{\alpha} c(\widetilde{E}_{\beta}) c(\widetilde{E}_s) c(\widetilde{E}_{\alpha}). \tag{3.11}$$

Since Φ is a global form on ∂M , so for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates U of x_0 in ∂M (not in M) and compute $\Phi(x_0)$ in the coordinates $\widetilde{U} = U \times [0, 1)$ and the metric $\frac{1}{h(x_n)} g^{\partial M} + \mathbf{d}x_n^2$. The dual metric of g^M on \widetilde{U} is $h(x_n) g^{\partial M} + \mathbf{d}x_n^2$. Write $g_{ij}^M = g^M(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$; $g_M^{ij} = g^M(dx_i, dx_j)$, then

$$[g_{i,j}^M] = \begin{bmatrix} \frac{1}{h(x_n)} [g_{i,j}^{\partial M}] & 0 \\ 0 & 1 \end{bmatrix}; \quad [g_M^{i,j}] = \begin{bmatrix} h(x_n) [g_{\partial M}^{i,j}] & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.12}$$

and

$$\partial_{x_s} g_{ij}^{\partial M}(x_0) = 0, \quad 1 \leq i, j \leq n-1; \quad g_{i,j}^M(x_0) = \delta_{ij}. \tag{3.13}$$

Let $\{E_1, \dots, E_{n-1}\}$ be an orthonormal frame field in U about $g^{\partial M}$ which is parallel along geodesics and $E_i = \frac{\partial}{\partial x_i}(x_0)$, then $\{\widetilde{E}_1 = \sqrt{h(x_n)} E_1, \dots, \widetilde{E}_{n-1} = \sqrt{h(x_n)} E_{n-1}, \widetilde{E}_n = dx_n\}$ is the orthonormal frame field in \widetilde{U} about g^M . Locally $S(TM)|_{\widetilde{U}} \cong \widetilde{U} \times \wedge_C^*(\frac{n}{2})$. Let $\{\widetilde{f}_1, \dots, \widetilde{f}_n\}$ be the orthonormal basis of $\wedge_C^*(\frac{n}{2})$. Take a spin frame field $\sigma: \widetilde{U} \rightarrow Spin(M)$ such that $\pi\sigma = \{\widetilde{E}_1, \dots, \widetilde{E}_n\}$ where $\pi: Spin(M) \rightarrow O(M)$ is a double covering, then $\{[\sigma, f_i], 1 \leq i \leq 4\}$ is an orthonormal frame of $S(TM)|_{\widetilde{U}}$. In the following, since the global form Φ is independent of the choice of the local frame, so we can compute $\text{tr}_{S(TM)}$ in the frame $\{[\sigma, f_i], 1 \leq i \leq 4\}$. Let $\{\hat{E}_1, \dots, \hat{E}_n\}$ be the canonical basis of R^n and $c(\hat{E}_i) \in cl_C(n) \cong Hom(\wedge_C^*(\frac{n}{2}), \wedge_C^*(\frac{n}{2}))$ be the Clifford action. By [24], then

$$c(\widetilde{E}_i) = [(\sigma, c(\hat{E}_i))]; \quad c(\widetilde{E}_i)[(\sigma, f_i)] = [\sigma, (c(\hat{E}_i))f_i]; \quad \frac{\partial}{\partial x_i} = [(\sigma, \frac{\partial}{\partial x_i})], \quad (3.14)$$

then we have $\frac{\partial}{\partial x_i} c(\widetilde{E}_i) = 0$ in the above frame. By Lemma 2.2 in [13], we have

Lemma 3.3. *With the metric g^M on M near the boundary*

$$\partial_{x_j}(|\xi|_{g^M}^2)(x_0) = \begin{cases} 0, & \text{if } j < n; \\ h'(0)|\xi'|_{g^{\partial M}}^2, & \text{if } j = n. \end{cases} \quad (3.15)$$

$$\partial_{x_j}[c(\xi)](x_0) = \begin{cases} 0, & \text{if } j < n; \\ \partial_{x_n}(c(\xi'))(x_0), & \text{if } j = n, \end{cases} \quad (3.16)$$

where $\xi = \xi' + \xi_n dx_n$.

Then

Lemma 3.4. *With the metric g^M on M near the boundary*

$$\partial_{x_i} \partial_{x_j}(|\xi|_{g^M}^2)(x_0) \Big|_{|\xi'|=1} = \begin{cases} 0, & \text{if } i < n, j = n; \text{ or } i = n, j < n; \\ -\frac{1}{3} \sum_{\alpha, \beta < n} \left(R_{i\alpha j\beta}^{\partial M}(x_0) + R_{i\beta j\alpha}^{\partial M}(x_0) \right) \xi_\alpha \xi_\beta, & \text{if } i, j < n; \\ h''(0), & \text{if } i = j = n. \end{cases} \quad (3.17)$$

$$\partial_{x_i} \partial_{x_j}[c(\xi)](x_0) \Big|_{|\xi'|=1} = \begin{cases} 0, & \text{if } i < n, j = n; \text{ or } i = n, j < n; \\ \frac{1}{6} \sum_{l, t < n} \xi_l \left(R_{tilj}^{\partial M}(x_0) + R_{tjli}^{\partial M}(x_0) \right) c(\widetilde{E}_t), & \text{if } i, j < n; \\ \left(\frac{3}{4}(h'(0))^2 - \frac{1}{2}h''(0) \right) \sum_{j < n} \xi_j c(\widetilde{E}_j), & \text{if } i = j = n, \end{cases} \quad (3.18)$$

where $\xi = \xi' + \xi_n dx_n$.

Proof. From proposition 1.28 in [23], we have

$$g_{ij}(x) \sim \delta_{ij} - \frac{1}{3} \sum_{kl} R_{ikjl}^{\partial M}(x_0) x^k x^l + \sum_{|\alpha| \geq 3} (\partial^\alpha g_{ij})(x_0) \frac{x^\alpha}{\alpha!}. \quad (3.19)$$

When $i, j < n$, we obtain

$$\begin{aligned} \partial_{x_i} \partial_{x_j}(|\xi|_{g^M}^2)(x_0) &= \partial_{x_i} \partial_{x_j} \left(h(x_n) |\xi'|^2 + \xi_n^2 \right)(x_0) \\ &= \partial_{x_i} \partial_{x_j} (|\xi'|^2)(x_0) \\ &= \partial_{x_i} \partial_{x_j} (g_{\alpha\beta} \xi_\alpha \xi_\beta)(x_0) \\ &= \partial_{x_i} \partial_{x_j} \left(\delta_{\alpha\beta} - \frac{1}{3} \sum_{kl} R_{\alpha k \beta l}^{\partial M}(x_0) x^k x^l + \dots \right)(x_0) \xi_\alpha \xi_\beta \\ &= -\frac{1}{3} \sum_{\alpha, \beta < n} \left(R_{i\alpha j\beta}^{\partial M}(x_0) + R_{i\beta j\alpha}^{\partial M}(x_0) \right) \xi_\alpha \xi_\beta. \end{aligned} \quad (3.20)$$

When $i < n, j = n$; or $i = n, j < n$, from lemma 3.3, we get $\partial_{x_i} \partial_{x_j}(|\xi|_{g^M}^2)(x_0) = 0$. When $i = j = n$, then $\partial_{x_n} \partial_{x_n}(|\xi|_{g^M}^2)(x_0) \Big|_{|\xi'|=1} = \partial_{x_n} \partial_{x_n} \left(h(x_n) |\xi'|^2 + \xi_n^2 \right)(x_0) \Big|_{|\xi'|=1} = h''(0)$.

On the other hand, let $H_{ij} = \omega_i(\frac{\partial}{\partial x_j})$, from lemma 1.7.10 in [24], we have

$$H_{ij}(x) \sim \delta_{ij} - \frac{1}{6} \sum_{kl} R_{ikjl}^{\partial_M}(x_0) x^k x^l + \dots \quad (3.21)$$

When $i, j < n$, we obtain

$$\begin{aligned} \partial_{x_i} \partial_{x_j} (c(\xi))(x_0) &= \partial_{x_i} \partial_{x_j} \left(c(\xi') + \xi_n c(\mathbf{d}x_n) \right) (x_0) \\ &= \partial_{x_i} \partial_{x_j} \left(\sum_l \xi_l c(\mathbf{d}x_l) \right) (x_0) \\ &= \partial_{x_i} \partial_{x_j} \left(\sum_{l,s,t} \xi_l (g^{sl} H_{st}) c(\tilde{E}_t) \right) (x_0) \\ &= \sum_{l,t} \xi_l \left(\partial_{x_i} \partial_{x_j} g^{sl} \delta_s^t + \delta_s^l \partial_{x_i} \partial_{x_j} H_{st} \right) (x_0) c(\tilde{E}_t) \\ &= \sum_{l,t} \xi_l \left(-\partial_{x_i} \partial_{x_j} g_{sl} \delta_s^t + \delta_s^l \partial_{x_i} \partial_{x_j} H_{st} \right) (x_0) c(\tilde{E}_t) \\ &= -\sum_{l,t} \xi_l \partial_{x_i} \partial_{x_j} \left(\delta_{lt} - \frac{1}{3} \sum_{\alpha\beta} R_{t\alpha l\beta}^{\partial_M}(x_0) x^\alpha x^\beta + \dots \right) (x_0) c(\tilde{E}_t) \\ &\quad + \sum_{l,t} \xi_l \partial_{x_i} \partial_{x_j} \left(\delta_{lt} - \frac{1}{6} \sum_{kl} R_{l\alpha t\beta}^{\partial_M}(x_0) x^\alpha x^\beta + \dots \right) (x_0) c(\tilde{E}_t) \\ &= \frac{1}{6} \sum_{l,t < n} \xi_l \left(R_{tilj}^{\partial_M}(x_0) + R_{tjli}^{\partial_M}(x_0) \right) c(\tilde{E}_t), \end{aligned} \quad (3.22)$$

where we have used the following fact of lemma A.1 in [13]

$$c(dx_j) = \sum_{1 \leq i, s < n} \frac{1}{\sqrt{h(x_n)}} g^{s,j} H_{s,i} c(\tilde{E}_i) + \sum_{i=s=n} g^{n,j} c(\tilde{E}_n). \quad (3.23)$$

If $i < n, j = n$; or $i = n, j < n$, from Lemma 3.3, we get $\partial_{x_i} \partial_{x_j} (c(\xi))(x_0) = 0$. When $i = j = n$, then

$$\partial_{x_n} \partial_{x_n} (c(\xi))(x_0) = \sum_{j < n} \xi_j \left(\frac{1}{\sqrt{h(x_n)}} \right)'' c(\tilde{E}_j) = \left(\frac{3}{4} (h'(0))^2 - \frac{1}{2} h''(0) \right) \sum_j \xi_j c(\tilde{E}_j). \quad (3.24)$$

□

Then an application of Lemma 2.3 in [13] shows

Lemma 3.5. *The following identity holds:*

$$p_0(x_0) = -h'(0) c(\mathbf{d}x_n); \quad (3.25)$$

$$\partial_{x_i} [A_{\beta s}^\alpha](x_0) = \begin{cases} \sum_{\beta, i, s, \alpha} R_{\beta i s \alpha}^{\partial_M}(x_0), & \text{if } i < n; \\ 0, & \text{if } i = n. \end{cases} \quad (3.26)$$

Proof. From lemma 5.7 in [25], we have

$$A_{\beta s}^\alpha = R_{\beta l s \alpha} x_l + O(|x|^2). \quad (3.27)$$

Then we obtain $\partial_{x_i} [A_{\beta s}^\alpha](x_0) = \sum_{\beta i s \alpha} R_{\beta i s \alpha}^{\partial_M}(x_0)$.

□

Lemma 3.6. When $i < n$,

$$\begin{aligned}
\partial_{x_i} \left(\sigma_{-2}(D^{-1}) \right) (x_0) \Big|_{|\xi'|=1} &= \frac{1}{8} \sum_{\beta i s \alpha} R_{\beta i s \alpha}^{\partial_M}(x_0) \frac{c(\xi) c(\tilde{E}_\beta) c(\tilde{E}_s) c(\tilde{E}_\alpha) c(\xi)}{(1 + \xi_n^2)^2} \\
&+ \frac{1}{6} \sum_{l, t < n} \xi_l \left(R_{t i l j}^{\partial_M}(x_0) + R_{t j l i}^{\partial_M}(x_0) \right) \frac{c(\xi) c(\mathbf{d}x_j) c(\tilde{E}_t)}{(1 + \xi_n^2)^2} \\
&+ \frac{1}{3} \sum_{\alpha, \beta < n} \left(R_{i \alpha j \beta}^{\partial_M}(x_0) + R_{i \beta j \alpha}^{\partial_M}(x_0) \right) \xi_\alpha \xi_\beta \frac{c(\xi) c(\mathbf{d}x_j) c(\xi)}{(1 + \xi_n^2)^3}. \quad (3.28)
\end{aligned}$$

When $i = n$,

$$\begin{aligned}
\partial_{x_n} \left(\sigma_{-2}(D^{-1}) \right) (x_0) \Big|_{|\xi'|=1} &= \left(\frac{-h'}{(1 + \xi_n^2)^2} + \frac{-h'}{(1 + \xi_n^2)^3} \right) \partial_{x_n} [c(\xi')](x_0) c(\mathbf{d}x_n) c(\xi) \\
&+ \left(\frac{(h')^2 - h''}{(1 + \xi_n^2)^2} + \frac{2(h')^2 - h''}{(1 + \xi_n^2)^3} + \frac{3(h')^2}{(1 + \xi_n^2)^4} \right) c(\xi) c(\mathbf{d}x_n) c(\xi) \\
&+ \left(\frac{-h'}{(1 + \xi_n^2)^2} + \frac{-3h'}{(1 + \xi_n^2)^3} \right) c(\xi) c(\mathbf{d}x_n) \partial_{x_n} [c(\xi')](x_0) \\
&+ \frac{1}{(1 + \xi_n^2)^2} \partial_{x_n} [c(\xi')](x_0) c(\mathbf{d}x_n) \partial_{x_n} [c(\xi')](x_0) \\
&+ \left(\frac{3}{4} (h'(0))^2 - \frac{1}{2} h''(0) \right) \frac{1}{(1 + \xi_n^2)^3} c(\xi) c(\mathbf{d}x_n) c(\xi'). \quad (3.29)
\end{aligned}$$

Proof. When $i < n$, from lemma 3.3 and $\partial_{x_i}(c(\mathbf{d}x_j))(x_0)=0$, we get

$$\begin{aligned}
\partial_{x_i} \left(\sigma_{-2}(D^{-1}) \right) (x_0) &= \frac{1}{8} \sum_{\beta i s \alpha} \partial_{x_i} [A_{\beta s}^\alpha](x_0) \frac{c(\xi) c(\tilde{E}_\beta) c(\tilde{E}_s) c(\tilde{E}_\alpha) c(\xi)}{(1 + \xi_n^2)^2} \\
&+ \sum_{j < n} c(\xi) c(\mathbf{d}x_j) \partial_{x_i} \partial_{x_j} (c(\xi))(x_0) \frac{1}{(1 + \xi_n^2)^2} \\
&- \sum_{j < n} c(\xi) c(\mathbf{d}x_j) c(\xi) \partial_{x_i} \partial_{x_j} (|\xi|_{g^M}^2)(x_0) \frac{1}{(1 + \xi_n^2)^3}. \quad (3.30)
\end{aligned}$$

By lemma 3.4, we obtain (3.28). Similarly, the conclusion (3.29) then follows easily. \square

Let us now consider the q_{-3} . From (3.8) and the proof of lemma 3.6, we know that

Lemma 3.7. *The following identity holds:*

$$\begin{aligned}
q_{-3}(x_0) \Big|_{|\xi'|=1} &= \frac{-i(h')^2}{(1+\xi_n^2)^3} c(\xi) c(\mathbf{d}x_n) c(\xi) c(\mathbf{d}x_n) c(\xi) \\
&+ \frac{ih'}{(1+\xi_n^2)^3} c(\xi) c(\mathbf{d}x_n) c(\xi) c(\mathbf{d}x_n) \partial_{x_n} [c(\xi')](x_0) \\
&+ \frac{-i(h')^2}{(1+\xi_n^2)^4} c(\xi) c(\mathbf{d}x_n) c(\xi) c(\mathbf{d}x_n) c(\xi) \\
&+ \frac{1}{8} \sum_{\beta \text{ is } \alpha} R_{\beta \text{ is } \alpha}^{\partial_M}(x_0) \frac{-i}{(1+\xi_n^2)^3} c(\xi) c(\tilde{E}_i) c(\xi) c(\tilde{E}_\beta) c(\tilde{E}_s) c(\tilde{E}_\alpha) c(\xi) \\
&+ \frac{1}{6} \sum_{l, t < n} \xi_l \left(R_{tilj}^{\partial_M}(x_0) + R_{tjli}^{\partial_M}(x_0) \right) \frac{-i}{(1+\xi_n^2)^3} c(\xi) c(\tilde{E}_i) c(\xi) c(\mathbf{d}x_j) c(\tilde{E}_t) \\
&+ \frac{1}{3} \sum_{\alpha, \beta < n} \left(R_{i\alpha j\beta}^{\partial_M}(x_0) + R_{i\beta j\alpha}^{\partial_M}(x_0) \right) \xi_\alpha \xi_\beta \frac{-i}{(1+\xi_n^2)^4} c(\xi) c(\tilde{E}_i) c(\xi) c(\mathbf{d}x_j) c(\xi) \\
&+ i \left(\frac{h'}{(1+\xi_n^2)^3} + \frac{h'}{(1+\xi_n^2)^4} \right) c(\xi) c(\mathbf{d}x_n) \partial_{x_n} [c(\xi')](x_0) c(\mathbf{d}x_n) c(\xi) \\
&- i \left(\frac{(h')^2 - h''}{(1+\xi_n^2)^3} + \frac{2(h')^2 - h''}{(1+\xi_n^2)^4} + \frac{3(h')^2}{(1+\xi_n^2)^5} \right) c(\xi) c(\mathbf{d}x_n) c(\xi) c(\mathbf{d}x_n) c(\xi) \\
&+ i \left(\frac{h'}{(1+\xi_n^2)^3} + \frac{3h'}{(1+\xi_n^2)^4} \right) c(\xi) c(\mathbf{d}x_n) c(\xi) c(\mathbf{d}x_n) \partial_{x_n} [c(\xi')](x_0) \\
&+ \frac{-i}{(1+\xi_n^2)^3} c(\xi) c(\mathbf{d}x_n) \partial_{x_n} [c(\xi')](x_0) c(\mathbf{d}x_n) \partial_{x_n} [c(\xi')](x_0) \\
&+ \left(\frac{3}{4} (h'(0))^2 - \frac{1}{2} h''(0) \right) \frac{-i}{(1+\xi_n^2)^4} c(\xi) c(\mathbf{d}x_n) c(\xi) c(\mathbf{d}x_n) c(\xi'). \tag{3.31}
\end{aligned}$$

Lemma 3.8. *The following identity holds:*

$$\text{tr} \left[\partial_{x_n} c(\xi') \times \partial_{x_n} c(\xi') \right] (x_0) \Big|_{|\xi'|=1} = - (h'(0))^2. \tag{3.32}$$

Proof. Let $\xi' = \sum_{j=1}^4 \xi_j \mathbf{d}x^j$, then

$$c(\xi') = \sum_{j=1}^4 \xi_j c(\mathbf{d}x^j) = \sum_{j,l=1}^4 \xi_j \langle \mathbf{d}x^j, \tilde{E}^l \rangle c(\tilde{E}_l). \tag{3.33}$$

Set

$$c(\hat{\xi}) = \partial_{x_n} (c(\xi')) = \sum_{j,l=1}^4 \xi_j \partial_{x_n} \left(\sqrt{h} \langle \mathbf{d}x^j, E^l \rangle_{\partial_M} \right) c(\tilde{E}_l). \tag{3.34}$$

Then

$$\text{tr} \left[\partial_{x_n} c(\xi') \times \partial_{x_n} c(\xi') \right] = \text{tr} \left[c(\hat{\xi}) c(\hat{\xi}) \right] (x_0) = -|\hat{\xi}'|^2 \text{tr}(\text{id}) = -4|\hat{\xi}'|^2. \tag{3.35}$$

Where

$$\begin{aligned}
|\hat{\xi}'|^2(x_0) \Big|_{|\xi'|=1} &= \sum_{l=1}^4 \left[\sum_{j=1}^4 \xi_j \partial_{x_n} (\sqrt{h} \langle \mathbf{d}x^j, E^l \rangle_{\partial_M}) \right]^2 (x_0) \Big|_{|\xi'|=1} \\
&= \left(\partial_{x_n} (\sqrt{h}) \right)^2 |\xi'|^2(x_0) \Big|_{|\xi'|=1} \\
&= \frac{(h'(x_n))^2}{4h(x_n)}(x_0) = \frac{(h'(0))^2}{4}. \tag{3.36}
\end{aligned}$$

Substituting (3.36) into (3.35), we have proved this lemma. \square

From the remark above, now we can compute Φ (see formula (2.20) for definition of Φ). Since the sum is taken over $-r - \ell + k + j + |\alpha| = 4$, $r, \ell \leq -1$, then we have the $\int_{\partial_M} \Phi$ is the sum of the following fifteen cases:

Case (1): $r = -1, \ell = -1, k = 0, j = 1, |\alpha| = 1$

From (2.20), we have

$$\text{Case (1)} = \frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[\partial_{x_n} \partial_{\xi'}^{\alpha} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{x'}^{\alpha} \partial_{\xi_n}^2 \sigma_{-1}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.37)$$

By Lemma 3.3, for $i < n$, we have

$$\partial_{x_i} \sigma_{-1}(D^{-1})(x_0) = \partial_{x_i} \left(\frac{\sqrt{-1}c(\xi)}{|\xi|^2} \right) (x_0) = \frac{\sqrt{-1}\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2} - \frac{\sqrt{-1}c(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0. \quad (3.38)$$

So Case (1) vanishes.

Case (2): $r = -1, \ell = -1, k = 0, j = 2, |\alpha| = 0$

From (2.20), we have

$$\text{Case (2)} = \frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j=2} \text{trace} \left[\partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{\xi_n}^3 \sigma_{-1}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.39)$$

By Lemma 3.2, a simple computation shows

$$\partial_{\xi_n}^3 \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} = \frac{24\xi_n - 24\xi_n^3}{(1 + \xi_n^2)^4} \sqrt{-1}c(\xi') + \frac{-6\xi_n^4 + 36\xi_n^2 - 6}{(1 + \xi_n^2)^4} \sqrt{-1}c(dx_n), \quad (3.40)$$

and

$$\partial_{x_n} \sigma_{-1}(D^{-1}) = \frac{\sqrt{-1}\partial_{x_n}(c(\xi))}{|\xi|^2} - \frac{\sqrt{-1}c(\xi)\partial_{x_n}(|\xi|^2)}{|\xi|^4}. \quad (3.41)$$

From Lemma 3.2, Lemma 3.3 and Lemma 3.4, we obtain

$$\begin{aligned} \partial_{x_n}^2 \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} &= \partial_{x_n} \left(\frac{\sqrt{-1}\partial_{x_n}(c(\xi))}{|\xi|^2} - \frac{\sqrt{-1}c(\xi)\partial_{x_n}(|\xi|^2)}{|\xi|^4} \right) (x_0) \Big|_{|\xi'|=1} \\ &= \left\{ \frac{\sqrt{-1}\partial_{x_n}^2(c(\xi))}{|\xi|^2} - \frac{\sqrt{-1}\partial_{x_n}(c(\xi))\partial_{x_n}(|\xi|^2)}{|\xi|^4} \right. \\ &\quad \left. - \frac{\sqrt{-1}(\partial_{x_n}(c(\xi))\partial_{x_n}(|\xi|^2) + c(\xi)\partial_{x_n}^2(|\xi|^2))}{|\xi|^4} \right. \\ &\quad \left. + \frac{2\sqrt{-1}c(\xi)\partial_{x_n}(|\xi|^2)\partial_{x_n}(|\xi|^2)}{|\xi|^6} \right\} (x_0) \Big|_{|\xi'|=1} \\ &= \frac{\sqrt{-1} \left(\frac{3}{4}(h'(0))^2 - \frac{1}{2}h''(0) \right) c(\xi') - h''(0)c(\xi) - 2h'\partial_{x_n}(c(\xi'))}{(1 + \xi_n^2)^2} \\ &\quad + \sqrt{-1} \frac{2(h'(0))^2 c(\xi)}{(1 + \xi_n^2)^3}. \end{aligned} \quad (3.42)$$

By (2.15) and the Cauchy integral formula, then

$$\begin{aligned}
\pi_{\xi_n}^+ \left[\frac{c(\xi)}{(1 + \xi_n^2)^2} \right] &= \pi_{\xi_n}^+ \left[\frac{c(\xi') + \xi_n c(dx_n)}{(1 + \xi_n^2)^2} \right] \\
&= \frac{1}{2\pi i} \lim_{u \rightarrow 0^-} \int_{\Gamma^+} \frac{\frac{c(\xi') + \eta_n c(dx_n)}{(\eta_n + i)^2 (\xi_n + i u - \eta_n)}}{(\eta_n - i)^2} d\eta_n \\
&= \left[\frac{c(\xi') + \eta_n c(dx_n)}{(\eta_n + i)^2 (\xi_n - \eta_n)} \right]^{(1)} \Big|_{\eta_n = i} \\
&= -\frac{ic(\xi')}{4(\xi_n - i)} - \frac{c(\xi') + ic(dx_n)}{4(\xi_n - i)^2}.
\end{aligned} \tag{3.43}$$

Similarly, we obtain

$$\pi_{\xi_n}^+ \left[\frac{1}{(1 + \xi_n^2)^2} \right] = \frac{-2 - i\xi_n}{4(\xi_n - i)^2}, \quad \pi_{\xi_n}^+ \left[\frac{c(\xi)}{(1 + \xi_n^2)^3} \right] = \frac{-3i\xi_n^2 - 9\xi_n + 8i}{16(\xi_n - i)^3} c(\xi') + \frac{-i\xi_n - 3}{16(\xi_n - i)^3} c(dx_n). \tag{3.44}$$

From the remark above, it is easy to see

$$\begin{aligned}
\partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} &= \left(\frac{3}{4}(h'(0))^2 - \frac{1}{2}h''(0) \right) \frac{c(\xi')}{2(\xi_n - i)} - h'(0) \frac{\xi_n - 2i}{2(\xi_n - i)^2} \partial_{x_n}(c(\xi')) \\
&\quad - h''(0) \left[\frac{\xi_n - 2i}{4(\xi_n - i)^2} c(\xi') + \frac{1}{4(\xi_n - i)^2} c(dx_n) \right] \\
&\quad + 2i(h'(0))^2 \left[\frac{-3i\xi_n^2 - 9\xi_n + 8i}{16(\xi_n - i)^3} c(\xi') + \frac{-i\xi_n - 3}{16(\xi_n - i)^3} c(dx_n) \right].
\end{aligned} \tag{3.45}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then

$$\begin{aligned}
\text{tr}[c(\xi')c(dx_n)] &= 0; \quad \text{tr}[c(dx_n)^2] = -4; \quad \text{tr}[c(\xi')^2](x_0) \Big|_{|\xi'|=1} = -4; \\
\text{tr}[\partial_{x_n}[c(\xi')]c(dx_n)] &= 0; \quad \text{tr}[\partial_{x_n}c(\xi') \times c(\xi')](x_0) \Big|_{|\xi'|=1} = -2h'(0).
\end{aligned} \tag{3.46}$$

For more trace expansions, we can see [26]. From (3.40), (3.45) and (3.46) and direct computations, we obtain

$$\begin{aligned}
&\text{trace} \left[\partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{\xi_n}^3 \sigma_{-1}(D^{-1}) \right] (x_0) \Big|_{|\xi'|=1} \\
&= (h'(0))^2 \frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1 + \xi_n^2)^4} \\
&\quad + h''(0) \frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1 + \xi_n^2)^4}.
\end{aligned} \tag{3.47}$$

Therefore

$$\begin{aligned}
\text{Case (2)} &= -\frac{1}{6}(h'(0))^2 \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1 + \xi_n^2)^4} d\xi_n \sigma(\xi') dx' \\
&\quad - \frac{1}{6}h''(0) \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1 + \xi_n^2)^4} d\xi_n \sigma(\xi') dx' \\
&= -\frac{1}{6}(h'(0))^2 \Omega_3 \int_{\Gamma^+} \frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1 + \xi_n^2)^4} d\xi_n dx' \\
&\quad - \frac{1}{6}h''(0) \Omega_3 \int_{\Gamma^+} \frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1 + \xi_n^2)^4} d\xi_n dx' \\
&= -\frac{1}{6}(h'(0))^2 \frac{2\pi i}{6!} \left[\frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n + i)^4} \right]^{(6)} \Big|_{\xi_n=i} \Omega_3 dx' \\
&\quad - \frac{1}{6}h''(0) \frac{2\pi i}{5!} \left[\frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{(\xi_n + i)^4} \right]^{(5)} \Big|_{\xi_n=i} \Omega_3 dx' \\
&= \left(\frac{29}{64}(h'(0))^2 - \frac{3}{8}h''(0) \right) \pi \Omega_3 dx', \tag{3.48}
\end{aligned}$$

where Ω_3 is the canonical volume of S^3 .

Case (3): $r = -1$, $\ell = -1$, $k = 0$, $j = 0$, $|\alpha| = 2$

From (2.20), we have

$$\text{Case (3)} = \frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=2} \text{trace} \left[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \tag{3.49}$$

By Lemma 3.2, a simple computation shows

$$\begin{aligned}
\partial_{\xi'}^\alpha \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} &= \partial_{\xi_j} \partial_{\xi_i} \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} \\
&= \partial_{\xi_j} \left(\frac{\sqrt{-1}c(\mathbf{d}x_i)}{(1 + \xi_n^2)} - \frac{2\sqrt{-1}\xi_i c(\xi)}{(1 + \xi_n^2)^2} \right) \\
&= \sqrt{-1} \left(\frac{-2\xi_j c(\mathbf{d}x_i) - 2\delta_j^i c(\xi) - 2\xi_i c(\mathbf{d}x_j)}{(1 + \xi_n^2)^2} + \frac{8\xi_i \xi_j c(\xi)}{(1 + \xi_n^2)^3} \right). \tag{3.50}
\end{aligned}$$

By (3.43) and (3.44), we obtain

$$\begin{aligned}
\pi_{\xi_n}^+ \partial_{\xi'}^\alpha \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} &= \xi_j \frac{2i - \xi_n}{2(\xi_n - i)^2} c(\mathbf{d}x_i) + \frac{2i - \xi_n}{2(\xi_n - i)^2} c(\xi') \delta_j^i + \frac{-1}{2(\xi_n - i)^2} c(\mathbf{d}x_n) \delta_j^i \\
&\quad + \xi_i \frac{2i - \xi_n}{2(\xi_n - i)^2} c(\mathbf{d}x_j) + \xi_i \xi_j \frac{3\xi_n^2 - 9i\xi_n - 8}{2(\xi_n - i)^3} c(\xi') \\
&\quad + \xi_i \xi_j \frac{\xi_n - 3i}{2(\xi_n - i)^3} c(\mathbf{d}x_n). \tag{3.51}
\end{aligned}$$

On the other hand, by Lemma 3.2, Lemma 3.3 and Lemma 3.4, we obtain

$$\begin{aligned}
\partial_{x'}^\alpha \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} &= \partial_{x_i} \partial_{x_j} \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} \\
&= \partial_{x_i} \left(\frac{\sqrt{-1} \partial_{x_j}(c(\xi))}{|\xi|^2} - \frac{\sqrt{-1} c(\xi) \partial_{x_j}(|\xi|^2)}{|\xi|^4} \right) (x_0) \Big|_{|\xi'|=1} \\
&= \left(\frac{\sqrt{-1} \partial_{x_i} \partial_{x_j}(c(\xi))}{|\xi|^2} - \frac{\sqrt{-1} c(\xi) \partial_{x_i} \partial_{x_j}(|\xi|^2)}{|\xi|^4} \right) (x_0) \Big|_{|\xi'|=1} \\
&= \frac{1}{6} \sum_{l,t < n} \xi_l \left(R_{tilj}^{\partial_M}(x_0) + R_{tjli}^{\partial_M}(x_0) \right) \frac{i}{(1 + \xi_n^2)} c(\tilde{E}_t) \\
&\quad + \frac{1}{3} \sum_{\alpha, \beta < n} \left(R_{i\alpha j\beta}^{\partial_M}(x_0) + R_{i\beta j\alpha}^{\partial_M}(x_0) \right) \xi_\alpha \xi_\beta \frac{i}{(1 + \xi_n^2)^2} c(\xi). \tag{3.52}
\end{aligned}$$

Hence in this case,

$$\begin{aligned}
\partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} &= \sum_{l,t < n} \xi_l \left(R_{tilj}^{\partial_M}(x_0) + R_{tjli}^{\partial_M}(x_0) \right) \frac{-2i\xi_n}{(1 + \xi_n^2)^2} c(\tilde{E}_t) \\
&\quad + \sum_{\alpha, \beta < n} \left(R_{i\alpha j\beta}^{\partial_M}(x_0) + R_{i\beta j\alpha}^{\partial_M}(x_0) \right) \xi_\alpha \xi_\beta \frac{-4i\xi_n}{3(1 + \xi_n^2)^3} c(\xi') \\
&\quad + \sum_{\alpha, \beta < n} \left(R_{i\alpha j\beta}^{\partial_M}(x_0) + R_{i\beta j\alpha}^{\partial_M}(x_0) \right) \xi_\alpha \xi_\beta \frac{i - 3i\xi_n^2}{3(1 + \xi_n^2)^3} c(\mathbf{d}x_n). \tag{3.53}
\end{aligned}$$

Considering for $i < n$, $\int_{|\xi'|=1} \{\xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2d+1}}\} \sigma(\xi') = 0$. From (3.46), (3.51), (3.53) and direct computations, we obtain

$$\begin{aligned}
&\text{trace} \left[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}(D^{-1}) \right] (x_0) \\
&= \sum_{l,j,t < n} R_{tjlt}^{\partial_M}(x_0) \xi_l \xi_j \frac{-4i\xi_n^2 - 8\xi_n}{3(\xi_n - i)^2(1 + \xi_n^2)^2} \\
&\quad + \sum_{t,i,l < n} R_{tili}^{\partial_M}(x_0) \xi_l \xi_t \frac{-4i\xi_n^4 - 8\xi_n^3 - 32i\xi_n^2 - 40\xi_n + 4i}{3(\xi_n - i)^2(1 + \xi_n^2)^3}. \tag{3.54}
\end{aligned}$$

Then an application of (16) in [6] shows

$$\int \xi^\mu \xi^\nu = \frac{1}{4} [\mu\nu], \quad \int \xi^\mu \xi^\nu \xi^\alpha \xi^\beta = \frac{1}{3 \cdot 2^3} [\mu\nu\alpha\beta], \tag{3.55}$$

where $[\mu\nu\alpha\beta]$ stands for the sum of products of $g^{\alpha\beta}$ determined by all "pairings" of $\mu\nu\alpha\beta$. Using the integration over S^3 and the shorthand $\int = \frac{1}{\pi^2} \int_{S^3} d^3\nu$, we obtain $\Omega_3 = 2\pi^2$. Let s_{∂_M} is the scalar curvature ∂_M , then

$$\sum_{\alpha, i, s < n} R_{\alpha is\alpha}^{\partial_M}(x_0) \int_{|\xi'|=1} \xi_i \xi_s \sigma(\xi') = \sum_{\alpha, i, s < n} R_{\alpha is\alpha}^{\partial_M}(x_0) \frac{\pi^2}{2} \delta_i^s = -\frac{1}{4} s_{\partial_M} \Omega_3, \tag{3.56}$$

$$\sum_{i, \alpha, j, \beta < n} R_{i\alpha j\beta}^{\partial_M}(x_0) \int_{|\xi'|=1} \xi_\alpha \xi_\beta \xi_i \xi_j \sigma(\xi') = \frac{\pi^2}{2} \sum_{i, \alpha, j, \beta < n} R_{i\alpha j\beta}^{\partial_M}(x_0) \left(\delta_\alpha^\beta \delta_i^j + \delta_\alpha^i \delta_\beta^j + \delta_\alpha^j \delta_\beta^i \right) = 0. \tag{3.57}$$

Therefore

$$\begin{aligned}
\text{Case (3)} &= \frac{i}{2} \int_{|\xi'|=1} \sum_{l,j,t < n} R_{tjlt}^{\partial_M}(x_0) \xi_l \xi_j \sigma(\xi') \int_{-\infty}^{+\infty} \frac{-4i\xi_n^2 - 8\xi_n}{3(\xi_n - i)^2(1 + \xi_n^2)^2} d\xi_n dx' \\
&+ \frac{i}{2} \int_{|\xi'|=1} \sum_{t,i,l < n} R_{tili}^{\partial_M}(x_0) \xi_l \xi_t \sigma(\xi') \int_{-\infty}^{+\infty} \frac{-4i\xi_n^4 - 8\xi_n^3 - 32i\xi_n^2 - 40\xi_n + 4i}{3(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n dx' \\
&= \frac{i}{2} \sum_{l,j,t < n} R_{tjlt}^{\partial_M}(x_0) \frac{\pi^2}{2} \delta_l^j \frac{2\pi i}{3!} \left[\frac{-4i\xi_n^2 - 8\xi_n}{3(\xi_n + i)^2} \right]^{(3)} \Big|_{\xi_n=i} \Omega_3 dx' \\
&+ \frac{i}{2} \sum_{t,i,l < n} R_{tili}^{\partial_M}(x_0) \frac{\pi^2}{2} \delta_t^i \frac{2\pi i}{4!} \left[\frac{-4i\xi_n^4 - 8\xi_n^3 - 32i\xi_n^2 - 40\xi_n + 4i}{3(\xi_n + i)^3} \right]^{(4)} \Big|_{\xi_n=i} \Omega_3 dx' \\
&= \frac{i}{2} \left(\frac{-i}{6} \pi^3 \sum_{t,l < n} R_{tllt}^{\partial_M}(x_0) + \frac{-2i}{3} \pi^3 \sum_{t,l < n} R_{tlll}^{\partial_M}(x_0) \right) dx' \\
&= \frac{1}{4} s_{\partial_M} \pi^3 dx', \tag{3.58}
\end{aligned}$$

where $\sum_{t,l < n} R_{tllt}^{\partial_M}(x_0)$ is the scalar curvature s_{∂_M} .

Case (4): $r = -1$, $\ell = -1$, $k = 1$, $j = 1$, $|\alpha| = 0$

From (2.20) and the Leibniz rule, we obtain

$$\begin{aligned}
\text{Case (4)} &= \frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{x_n} \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
&= -\frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{\xi_n}^3 \partial_{x_n} \sigma_{-1}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \tag{3.59}
\end{aligned}$$

By (2.2.22) in [13], we have

$$\pi_{\xi_n}^+ \partial_{x_n} \sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1} = \frac{\partial_{x_n} [c(\xi')](x_0)}{2(\xi_n - i)} + \sqrt{-1} h'(0) \left[\frac{ic(\xi')}{4(\xi_n - i)} + \frac{c(\xi') + ic(dx_n)}{4(\xi_n - i)^2} \right]. \tag{3.60}$$

From (3.42) and direct computations, we obtain

$$\begin{aligned}
\partial_{\xi_n}^3 \partial_{x_n} \sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1} &= \frac{24i\xi_n - 24i\xi_n^3}{(1 + \xi_n^2)^4} \partial_{x_n} [c(\xi')](x_0) + \sqrt{-1} h'(0) \left[\frac{8(15\xi_n^3 - 9\xi_n)}{(1 + \xi_n^2)^5} c(\xi') \right. \\
&\quad \left. + \frac{12(5\xi_n^4 - 10\xi_n^2 + 1)}{(1 + \xi_n^2)^5} c(dx_n) \right]. \tag{3.61}
\end{aligned}$$

From Lemma 3.8, combining (3.46), (3.59) and (3.60), we obtain

$$\begin{aligned}
&\text{trace} \left[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{\xi_n}^3 \partial_{x_n} \sigma_{-1}(D^{-1}) \right] (x_0) \\
&= (h'(0))^2 \frac{12(-\xi_n^5 + 5i\xi_n^4 + 10\xi_n^3 - 10i\xi_n^2 - 5\xi_n + i)}{(\xi_n - i)^2(1 + \xi_n^2)^5}. \tag{3.62}
\end{aligned}$$

Therefore

$$\begin{aligned}
\text{Case (4)} &= \frac{i}{6} (h'(0))^2 \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{12(-\xi_n^5 + 5i\xi_n^4 + 10\xi_n^3 - 10i\xi_n^2 - 5\xi_n + i)}{(\xi_n - i)^2(1 + \xi_n^2)^5} d\xi_n \sigma(\xi') dx' \\
&= \frac{i}{6} (h'(0))^2 \frac{2\pi i}{6!} \left[\frac{12(-\xi_n^5 + 5i\xi_n^4 + 10\xi_n^3 - 10i\xi_n^2 - 5\xi_n + i)}{(\xi_n + i)^5} \right]^{(6)} \Big|_{\xi_n=i} \Omega_3 dx' \\
&= -\frac{5}{16} (h'(0))^2 \pi \Omega_3 dx'. \tag{3.63}
\end{aligned}$$

Case (5): $r = -1, \ell = -1, k = 1, j = 0, |\alpha| = 1$
From (2.20), we have

$$\text{Case (5)} = \frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[\partial_{\xi'}^{\alpha} \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{x'}^{\alpha} \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.64)$$

From Lemma 3.3 and Lemma 3.4, for $i < n$, we have

$$\begin{aligned} \partial_{x_i} \partial_{x_n} \sigma_{-1}(D^{-1})(x_0) &= \partial_{x_i} \left[\partial_{x_n} \left(\frac{\sqrt{-1}c(\xi)}{|\xi|^2} \right) \right] (x_0) \\ &= \partial_{x_i} \left[\frac{\sqrt{-1}\partial_{x_n}(c(\xi))}{|\xi|^2} - \frac{\sqrt{-1}c(\xi)\partial_{x_n}(|\xi|^2)}{|\xi|^4} \right] (x_0) \\ &= \sqrt{-1} \left[\frac{\partial_{x_i}\partial_{x_n}(c(\xi))}{|\xi|^2} - \frac{\partial_{x_n}[c(\xi)]\partial_{x_i}(|\xi|^2)}{|\xi|^4} \right] (x_0) \\ &\quad - \sqrt{-1} \left[\frac{\partial_{x_i}(c(\xi))\partial_{x_n}(|\xi|^2)}{|\xi|^4} + \frac{c(\xi)\partial_{x_i}\partial_{x_n}(|\xi|^2)}{|\xi|^4} - \frac{2c(\xi)\partial_{x_n}(|\xi|^2)\partial_{x_i}(|\xi|^2)}{|\xi|^6} \right] (x_0) \\ &= 0. \end{aligned} \quad (3.65)$$

Therefore Case (5) vanishes.

Case (6): $r = -1, \ell = -1, k = 2, j = 0, |\alpha| = 0$
From (2.20), we have

$$\text{Case (6)} = \frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{k=2} \text{trace} \left[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{\xi_n} \partial_{x_n}^2 \sigma_{-1}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.66)$$

By the Leibniz rule, trace property and "++" and "--" vanishing after the integration over ξ_n in [9], then

$$\begin{aligned} &\int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{\xi_n} \partial_{x_n}^2 \sigma_{-1}(D^{-1}) \right] d\xi_n \\ &= - \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n}^3 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{x_n}^2 \sigma_{-1}(D^{-1}) \right] d\xi_n \\ &= - \left[\int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n}^3 \sigma_{-1}(D^{-1}) \partial_{x_n}^2 \sigma_{-1}(D^{-1}) \right] d\xi_n \right. \\ &\quad \left. - \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n}^3 \pi_{\xi_n}^- \sigma_{-1}(D^{-1}) \partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \right] d\xi_n \right] \\ &= - \left[\int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n}^3 \sigma_{-1}(D^{-1}) \partial_{x_n}^2 \sigma_{-1}(D^{-1}) \right] d\xi_n - \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n}^3 \sigma_{-1}(D^{-1}) \partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \right] d\xi_n \right] \\ &= \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n}^3 \sigma_{-1}(D^{-1}) \partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \right] d\xi_n - \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n}^3 \sigma_{-1}(D^{-1}) \partial_{x_n}^2 \sigma_{-1}(D^{-1}) \right] d\xi_n. \end{aligned} \quad (3.67)$$

Combining these assertions, we see

$$\text{Case (6)} = \text{Case (2)} - \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n}^3 \sigma_{-1}(D^{-1}) \partial_{x_n}^2 \sigma_{-1}(D^{-1}) \right] d\xi_n. \quad (3.68)$$

From Lemma 3.3, we have

$$\partial_{\xi_n}^3 \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} = \frac{24\xi_n - 24\xi_n^3}{(1 + \xi_n^2)^4} \sqrt{-1}c(\xi') + \frac{-6\xi_n^4 + 36\xi_n^2 - 6}{(1 + \xi_n^2)^4} \sqrt{-1}c(dx_n), \quad (3.69)$$

and

$$\begin{aligned} \partial_{x_n}^2 \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} &= \frac{-2\sqrt{-1}h'(0)}{(1+\xi_n^2)^2} \partial_{x_n}(c(\xi'))(x_0) + \frac{2\xi_n(h'(0))^2 - \xi_n(1+\xi_n^2)h''(x_0)}{(1+\xi_n^2)^3} \sqrt{-1}c(\mathbf{d}x_n) \\ &+ \frac{(11+3\xi_n^2)(h'(0))^2 - (6+6\xi_n^2)h''(x_0)}{4(1+\xi_n^2)^3} \sqrt{-1}c(\xi'). \end{aligned} \quad (3.70)$$

Combining (3.46), (3.69) and (3.70), we obtain

$$\text{trace} \left[\partial_{\xi_n}^3 \sigma_{-1}(D^{-1}) \partial_{x_n}^2 \sigma_{-1}(D^{-1}) \right] (x_0) = (h'(0))^2 \frac{24\xi_n(5-3\xi_n^2)}{(1+\xi_n^2)^5} + h''(0) \frac{24\xi_n(3\xi_n^2-5)}{(1+\xi_n^2)^5}. \quad (3.71)$$

We note that

$$\int_{-\infty}^{+\infty} \frac{24\xi_n(5-3\xi_n^2)}{(1+\xi_n^2)^5} \mathbf{d}\xi_n = \frac{2\pi i}{4!} \left[\frac{24\xi_n(5-3\xi_n^2)}{(\xi_n+i)^5} \right]^{(4)} \Big|_{\xi_n=i} = 0. \quad (3.72)$$

Therefore

$$\text{Case (6)} = \left(\frac{29}{64} (h'(0))^2 - \frac{3}{8} h''(0) \right) \pi \Omega_3 \mathbf{d}x'. \quad (3.73)$$

Case (7): $r = -1$, $\ell = -2$, $k = 0$, $j = 1$, $|\alpha| = 0$

From (2.20) and the Leibniz rule, we obtain

$$\begin{aligned} \text{Case (7)} &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n} \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{\xi_n} \sigma_{-2}(D^{-1}) \right] (x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x' \\ &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \sigma_{-2}(D^{-1}) \right] (x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x'. \end{aligned} \quad (3.74)$$

By Lemma 3.3 and (2.2.22) in [13], we have

$$\pi_{\xi_n}^+ \partial_{x_n} \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} = \frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n - i)} + \sqrt{-1}h'(0) \left[\frac{ic(\xi')}{4(\xi_n - i)} + \frac{c(\xi') + ic(\mathbf{d}x_n)}{4(\xi_n - i)^2} \right]. \quad (3.75)$$

By direct computations, we obtain

$$\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} = \frac{1}{(\xi - 1)^3} \partial_{x_n}[c(\xi')](x_0) + h'(0) \left[\frac{4i - \xi_n}{2(\xi - 1)^4} c(\xi') - \frac{3}{2(\xi - 1)^4} c(\mathbf{d}x_n) \right]. \quad (3.76)$$

From Lemma 3.2 and Lemma 3.5, we have

$$\sigma_{-2}(D^{-1})(x_0) = \frac{-h'(0)c(\xi)c(\mathbf{d}x_n)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(\mathbf{d}x_n) \left[\partial_{x_n}[c(\xi')](x_0)|\xi|^2 - c(\xi)h'(0)|\xi|_{2M}^2 \right]. \quad (3.77)$$

Considering for $i < n$, $\int_{|\xi'|=1} \{\xi_{i_1}\xi_{i_2} \cdots \xi_{i_{2d+1}}\} \sigma(\xi') = 0$. From Lemma 3.8, combining (3.46), (3.76), (3.77) and direct computations, we obtain

$$\text{trace} \left[\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \sigma_{-2}(D^{-1}) \right] (x_0) = (h'(0))^2 \frac{3(-2\xi_n^4 + 3i\xi_n^3 - 3\xi_n^2 + 7i\xi_n + 3)}{-(\xi_n - i)^4(1 + \xi_n^2)^3}. \quad (3.78)$$

Therefore

$$\begin{aligned} \text{Case (7)} &= \frac{1}{2} (h'(0))^2 \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{3(-2\xi_n^4 + 3i\xi_n^3 - 3\xi_n^2 + 7i\xi_n + 3)}{(\xi_n - i)^4(1 + \xi_n^2)^3} \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x' \\ &= \frac{1}{2} (h'(0))^2 \frac{2\pi i}{6!} \left[\frac{3(-2\xi_n^4 + 3i\xi_n^3 - 3\xi_n^2 + 7i\xi_n + 3)}{(\xi_n + i)^3} \right]^{(6)} \Big|_{\xi_n=i} \Omega_3 \mathbf{d}x' \\ &= \frac{39}{32} (h'(0))^2 \pi \Omega_3 \mathbf{d}x'. \end{aligned} \quad (3.79)$$

Case (8): $r = -1$, $\ell = -2$, $k = 0$, $j = 0$, $|\alpha| = 1$

From (2.20) and the Leibniz rule, we obtain

$$\begin{aligned} \text{Case (8)} &= - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-2}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\ &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[\partial_{\xi_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{x'}^\alpha \sigma_{-2}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \quad (3.80)$$

By Lemma 3.2, a simple computation shows

$$\begin{aligned} \partial_{\xi'}^\alpha \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} &= \partial_{\xi_i} \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} \\ &= \frac{\sqrt{-1} c(dx_i)}{(1 + \xi_n^2)} - \frac{2\sqrt{-1} \xi_i c(\xi)}{(1 + \xi_n^2)^2}. \end{aligned} \quad (3.81)$$

By (3.43) and (3.44), we obtain

$$\pi_{\xi_n}^+ \partial_{\xi'}^\alpha \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} = \frac{1}{2(\xi_n - i)^2} c(dx_i) - \xi_i \frac{\xi_n - 2i}{2(\xi_n - i)^2} c(\xi') - \xi_i \frac{1}{2(\xi_n - i)^2} c(dx_n). \quad (3.82)$$

Then

$$\partial_{\xi_n} \pi_{\xi_n}^+ \partial_{\xi'}^\alpha \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} = \frac{-1}{2(\xi_n - i)^2} c(dx_i) - \xi_i \frac{3i - \xi_n}{2(\xi_n - i)^3} c(\xi') + \xi_i \frac{1}{(\xi_n - i)^3} c(dx_n). \quad (3.83)$$

Considering for $i < n$, $\int_{|\xi'|=1} \{\xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2d+1}}\} \sigma(\xi') = 0$. By the relation of the Clifford action and $\text{tr} AB = \text{tr} BA$, then

$$\begin{aligned} & \sum_{\beta, i, s, \alpha} R_{\beta i s \alpha}^{\partial_M}(x_0) \text{tr} [c(dx_i) c(\xi) c(\tilde{E}_\beta) c(\tilde{E}_s) c(\tilde{E}_\alpha) c(\xi)] \\ &= \sum_{\beta, i, s, \alpha} R_{\beta i s \alpha}^{\partial_M}(x_0) \text{tr} [c(\xi) c(\tilde{E}_i) c(\xi) c(\tilde{E}_\beta) c(\tilde{E}_s) c(\tilde{E}_\alpha)] \\ &= \sum_{\beta, i, s, \alpha} R_{\beta i s \alpha}^{\partial_M}(x_0) \text{tr} [(-c(\tilde{E}_i) c(\xi) - 2\xi_i) c(\xi) c(\tilde{E}_\beta) c(\tilde{E}_s) c(\tilde{E}_\alpha)] \\ &= \sum_{\beta, i, s, \alpha} R_{\beta i s \alpha}^{\partial_M}(x_0) \text{tr} [(1 + \xi_n^2) c(\tilde{E}_i) c(\tilde{E}_\beta) c(\tilde{E}_s) c(\tilde{E}_\alpha)] - \sum_{\beta, i, s, \alpha} R_{\beta i s \alpha}^{\partial_M}(x_0) \text{tr} [2\xi_i c(\xi) c(\tilde{E}_\beta) c(\tilde{E}_s) c(\tilde{E}_\alpha)] \\ &= (1 + \xi_n^2) \sum_{\beta, i, s, \alpha} R_{\beta i s \alpha}^{\partial_M}(x_0) \left(-\delta_i^s \delta_\alpha^\beta + \delta_\alpha^i \delta_\beta^s \right) \text{tr} [\text{id}] \\ &\quad - 2 \sum_{\beta, i, s, \alpha} R_{\beta i s \alpha}^{\partial_M}(x_0) \xi_i \xi_\gamma \text{tr} [c(\tilde{E}_\gamma) c(\tilde{E}_\beta) c(\tilde{E}_s) c(\tilde{E}_\alpha)] \text{tr} [\text{id}] \\ &= (1 + \xi_n^2) \left(- \sum_{\beta, i} R_{\beta i i \beta}^{\partial_M}(x_0) + \sum_{\beta, i} R_{\beta i \beta i}^{\partial_M}(x_0) \right) \text{tr} [\text{id}] - 2 \sum_{\beta, i, s, \alpha} R_{\beta i s \alpha}^{\partial_M}(x_0) \xi_i \xi_\gamma \left(-\delta_\gamma^s \delta_\alpha^\beta + \delta_\alpha^\gamma \delta_\beta^s \right) \text{tr} [\text{id}] \\ &= 8(1 + \xi_n^2) s_{\partial_M} + 16 \sum_{i, s, \alpha} R_{\alpha i s \alpha}^{\partial_M}(x_0) \xi_i \xi_s. \end{aligned} \quad (3.84)$$

Similarly, we have

$$\sum_{i,l,t < n} \xi_l \left(R_{tilj}^{\partial_M}(x_0) + R_{tjli}^{\partial_M}(x_0) \right) \text{tr}[c(\mathbf{d}x_i)c(\xi)c(\mathbf{d}x_j)c(\tilde{E}_t)] = 16 \sum_{i,l,t < n} R_{tilt}^{\partial_M}(x_0) \xi_l \xi_i, \quad (3.85)$$

$$\sum_{\beta,i,s,\alpha} R_{\beta is\alpha}^{\partial_M}(x_0) \xi_i \text{tr}[c(\xi')c(\xi)c(\tilde{E}_\beta)c(\tilde{E}_s)c(\tilde{E}_\alpha)c(\xi)] = 8(\xi_n^2 - 1) \sum_{i,s,\alpha} R_{\alpha is\alpha}^{\partial_M}(x_0) \xi_i \xi_s, \quad (3.86)$$

$$\begin{aligned} & \sum_{i,\alpha,\beta < n} \left(R_{i\alpha j\beta}^{\partial_M}(x_0) + R_{i\beta j\alpha}^{\partial_M}(x_0) \right) \xi_i \xi_\alpha \xi_\beta \text{tr}[c(\xi')c(\xi)c(\mathbf{d}x_i)c(\xi)] \\ &= 8(1 - \xi_n^2) \sum_{i,j,\alpha,\beta < n} R_{i\alpha j\beta}^{\partial_M}(x_0) \xi_i \xi_j \xi_\alpha \xi_\beta. \end{aligned} \quad (3.87)$$

From (3.28), (3.56), (3.57), (3.82)-(3.87) and direct computations, we obtain

$$\begin{aligned} & \text{trace} \left[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}(D^{-1}) \right] (x_0) \\ &= s_{\partial_M} \frac{-1}{2(\xi_n - i)(1 + \xi_n^2)} + \sum_{\alpha,i,l < n} R_{\alpha il\alpha}^{\partial_M}(x_0) \xi_l \xi_i \frac{3\xi_n^3 - 9i\xi_n^2 + 22\xi_n^2 - 44i\xi_n - 21\xi_n - 22 + 15i}{-6(\xi_n - i)^3(1 + \xi_n^2)^2}. \end{aligned} \quad (3.88)$$

Then by (3.56) and (3.88) we have

$$\begin{aligned} \text{Case (8)} &= s_{\partial_M} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{-1}{2(\xi_n - i)(1 + \xi_n^2)} \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x' + \int_{|\xi'|=1} \sum_{\alpha,i,l < n} R_{\alpha il\alpha}^{\partial_M}(x_0) \xi_l \xi_i \sigma(\xi') \\ &\quad \times \int_{-\infty}^{+\infty} \frac{3\xi_n^3 - 9i\xi_n^2 + 22\xi_n^2 - 44i\xi_n - 21\xi_n - 22 + 15i}{-6(\xi_n - i)^3(1 + \xi_n^2)^2} \mathbf{d}\xi_n \mathbf{d}x' \\ &= s_{\partial_M} 2\pi i \left[\frac{-1}{2(\xi_n + i)} \right]^{(1)} \Big|_{\xi_n=i} \Omega_3 \mathbf{d}x' \\ &\quad + \sum_{i,l < n} R_{\alpha il\alpha}^{\partial_M}(x_0) \frac{\pi^2}{2} \delta_l^i \frac{2\pi i}{4!} \left[\frac{3\xi_n^3 - 9i\xi_n^2 + 22\xi_n^2 - 44i\xi_n - 21\xi_n - 22 + 15i}{-6(\xi_n + i)^2} \right]^{(4)} \Big|_{\xi_n=i} \mathbf{d}x' \\ &= \frac{-i}{4} s_{\partial_M} \pi \Omega_3 \mathbf{d}x' - \frac{1}{4} s_{\partial_M} \pi \Omega_3 \left(-\frac{3}{4} - \frac{11i}{8} \right) \mathbf{d}x' \\ &= \left(\frac{3}{16} + \frac{3}{32}i \right) s_{\partial_M} \pi \Omega_3 \mathbf{d}x'. \end{aligned} \quad (3.89)$$

Case (9): $r = -1, \ell = -2, k = 1, j = 0, |\alpha| = 0$

From (2.20) and the Leibniz rule, we obtain

$$\begin{aligned} \text{Case (9)} &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{\xi_n} \partial_{x_n} \sigma_{-2}(D^{-1}) \right] (x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x' \\ &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{x_n} \sigma_{-2}(D^{-1}) \right] (x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x'. \end{aligned} \quad (3.90)$$

By (2.2.29) in [13], we have

$$\partial_{\xi_n} \pi_{\xi_n}^+ q_{-1}(x_0) \Big|_{|\xi'|=1} = -\frac{c(\xi') + ic(\mathbf{d}x_n)}{2(\xi_n - i)^2}. \quad (3.91)$$

Then

$$\partial_{\xi_n}^2 \pi_{\xi_n}^+ q_{-1}(x_0) \Big|_{|\xi'|=1} = \frac{1}{(\xi_n - i)^3} c(\xi') + \frac{i}{2(\xi_n - i)^3} c(\mathbf{d}x_n). \quad (3.92)$$

Combining (3.29) and (3.92), we obtain

$$\begin{aligned}
& \text{trace} \left[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{x_n} \sigma_{-2}(D^{-1}) \right] (x_0) \Big|_{|\xi'|=1} \\
&= (h'(0))^2 \frac{4i\xi_n^6 + 4\xi_n^5 + 12i\xi_n^4 + 19\xi_n^3 + 13i\xi_n^2 + 39\xi_n - 19i}{(\xi_n - i)^3(1 + \xi_n^2)^4} \\
&\quad - h''(0) \frac{2(2i\xi_n^4 + 4\xi_n^3 + 2i\xi_n^2 + 9\xi_n - 5i)}{(\xi_n - i)^3(1 + \xi_n^2)^3}.
\end{aligned} \tag{3.93}$$

Therefore

$$\begin{aligned}
\text{Case (9)} &= \frac{1}{2} (h'(0))^2 \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{4i\xi_n^6 + 4\xi_n^5 + 12i\xi_n^4 + 19\xi_n^3 + 13i\xi_n^2 + 39\xi_n - 19i}{(\xi_n - i)^3(1 + \xi_n^2)^4} d\xi_n \sigma(\xi') dx' \\
&\quad - \frac{1}{2} h''(0) \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{2(2i\xi_n^4 + 4\xi_n^3 + 2i\xi_n^2 + 9\xi_n - 5i)}{(\xi_n - i)^3(1 + \xi_n^2)^3} d\xi_n \sigma(\xi') dx' \\
&= \frac{1}{2} (h'(0))^2 \Omega_3 \int_{\Gamma^+} \frac{4i\xi_n^6 + 4\xi_n^5 + 12i\xi_n^4 + 19\xi_n^3 + 13i\xi_n^2 + 39\xi_n - 19i}{(\xi_n - i)^3(1 + \xi_n^2)^4} d\xi_n dx' \\
&\quad - \frac{1}{2} h''(0) \Omega_3 \int_{\Gamma^+} \frac{2(2i\xi_n^4 + 4\xi_n^3 + 2i\xi_n^2 + 9\xi_n - 5i)}{(\xi_n - i)^3(1 + \xi_n^2)^3} d\xi_n dx' \\
&= \frac{1}{2} (h'(0))^2 \frac{2\pi i}{6!} \left[\frac{4i\xi_n^6 + 4\xi_n^5 + 12i\xi_n^4 + 19\xi_n^3 + 13i\xi_n^2 + 39\xi_n - 19i}{(\xi_n + i)^4} \right]^{(6)} \Big|_{\xi_n=i} \Omega_3 dx' \\
&\quad - \frac{1}{2} h''(0) \frac{2\pi i}{5!} \left[\frac{2(2i\xi_n^4 + 4\xi_n^3 + 2i\xi_n^2 + 9\xi_n - 5i)}{(\xi_n + i)^3} \right]^{(5)} \Big|_{\xi_n=i} \Omega_3 dx' \\
&= \left(-\frac{367}{128} (h'(0))^2 + \frac{103}{64} h''(0) \right) \pi \Omega_3 dx'.
\end{aligned} \tag{3.94}$$

Case (10): $r = -2$, $\ell = -1$, $k = 0$, $j = 1$, $|\alpha| = 0$

From (2.20), we have

$$\text{Case (10)} = \frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \partial_{\xi_n}^2 \sigma_{-1}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \tag{3.95}$$

By the Leibniz rule, trace property and "++" and "--" vanishing after the integration over ξ_n in [9], then

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \partial_{\xi_n}^2 \sigma_{-1}(D^{-1}) \right] d\xi_n \\
&= \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{x_n} \sigma_{-2}(D^{-1}) \partial_{\xi_n}^2 \sigma_{-1}(D^{-1}) \right] d\xi_n - \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{x_n} \sigma_{-2}(D^{-1}) \partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \right] d\xi_n.
\end{aligned} \tag{3.96}$$

Combining these assertions, we see

$$\text{Case (10)} = \text{Case (9)} - \frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{x_n} \sigma_{-2}(D^{-1}) \partial_{\xi_n}^2 \sigma_{-1}(D^{-1}) \right] d\xi_n \sigma(\xi') dx'. \tag{3.97}$$

By Lemma 3.2, a simple computation shows

$$\partial_{\xi_n}^2 \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} = \frac{6\xi_n^2 - 2}{(1 + \xi_n^2)^3} \sqrt{-1} c(\xi') + \frac{2\xi_n^3 - 6\xi_n}{(1 + \xi_n^2)^3} \sqrt{-1} c(dx_n). \tag{3.98}$$

Combining (3.29) and (3.98), we obtain

$$\text{trace}\left[\partial_{x_n}\sigma_{-2}(D^{-1})\partial_{\xi_n}^2\sigma_{-1}(D^{-1})\right](x_0) = (h'(0))^2\frac{8i\xi_n^5 + 8i\xi_n^3 + 36i\xi_n}{(1 + \xi_n^2)^5} + h''(0)\frac{8i\xi_n^5 + 24i\xi_n^3 + 24i\xi_n}{(1 + \xi_n^2)^5}. \quad (3.99)$$

We note that

$$\int_{-\infty}^{+\infty} \frac{8i\xi_n^5 + 8i\xi_n^3 + 36i\xi_n}{(1 + \xi_n^2)^5} d\xi_n = \frac{2\pi i}{4!} \left[\frac{8i\xi_n^5 + 8i\xi_n^3 + 36i\xi_n}{(\xi_n + i)^5} \right]^{(4)} \Big|_{\xi_n=i} = 0, \quad (3.100)$$

and

$$\int_{-\infty}^{+\infty} \frac{8i\xi_n^5 + 24i\xi_n^3 + 24i\xi_n}{(1 + \xi_n^2)^5} d\xi_n = 0. \quad (3.101)$$

Therefore

$$\text{Case (10)} = \left(-\frac{367}{128}(h'(0))^2 + \frac{103}{64}h''(0) \right) \pi\Omega_3 dx'. \quad (3.102)$$

Case (11): $r = -2, \ell = -1, k = 0, j = 0, |\alpha| = 1$

From (2.20), we have

$$\text{Case (11)} = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}\left[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}(D^{-1})\right](x_0) d\xi_n \sigma(\xi') dx'. \quad (3.103)$$

By Lemma 3.3, for $i < n$, we have

$$\partial_{x_i} \sigma_{-1}(D^{-1})(x_0) = \partial_{x_i} \left(\frac{\sqrt{-1}c(\xi)}{|\xi|^2} \right) (x_0) = \frac{\sqrt{-1}\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2} - \frac{\sqrt{-1}c(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0. \quad (3.104)$$

So Case (11) vanishes.

Case (12): $r = -2, \ell = -1, k = 1, j = 0, |\alpha| = 0$

From (2.20) and the Leibniz rule, we have

$$\begin{aligned} \text{Case (12)} &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}\left[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(D^{-1})\right](x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}\left[\pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}(D^{-1})\right](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \quad (3.105)$$

By the Leibniz rule, trace property and "++" and "--" vanishing after the integration over ξ_n in [9], then

$$\begin{aligned} &\int_{-\infty}^{+\infty} \text{trace}\left[\pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}(D^{-1})\right] d\xi_n \\ &= \int_{-\infty}^{+\infty} \text{trace}\left[\sigma_{-2}(D^{-1}) \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}(D^{-1})\right] d\xi_n - \int_{-\infty}^{+\infty} \text{trace}\left[\sigma_{-2}(D^{-1}) \partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1})\right] d\xi_n. \end{aligned} \quad (3.106)$$

Combining these assertions, we see

$$\text{Case (12)} = \text{Case (7)} + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}\left[\sigma_{-2}(D^{-1}) \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}(D^{-1})\right](x_0) d\xi_n \sigma(\xi') dx'. \quad (3.107)$$

From (3.42) and direct computations, we obtain

$$\begin{aligned} \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1} &= \frac{6i\xi_n^2 - 2i}{(1 + \xi_n^2)^3} \partial_{x_n}[c(\xi')](x_0) + \sqrt{-1}h'(0) \left[\frac{4(1 - 5\xi_n^2)}{(1 + \xi_n^2)^4} c(\xi') \right. \\ &\quad \left. - \frac{12\xi_n(\xi_n^2 - 1)}{(1 + \xi_n^2)^4} c(dx_n) \right]. \end{aligned} \quad (3.108)$$

From Lemma 3.8, combining (3.77) and (3.108), we obtain

$$\text{trace}\left[\sigma_{-2}(D^{-1})\partial_{\xi_n}^2\partial_{x_n}\sigma_{-1}(D^{-1})\right](x_0) = (h'(0))^2 \frac{30i\xi_n}{(1+\xi_n^2)^4}. \quad (3.109)$$

We note that

$$\int_{-\infty}^{+\infty} \frac{30i\xi_n}{(1+\xi_n^2)^4} d\xi_n = \frac{2\pi i}{3!} \left[\frac{30i\xi_n}{(\xi_n+i)^4} \right]^{(3)} \Big|_{\xi_n=i} = 0. \quad (3.110)$$

Therefore

$$\text{Case (12)} = \frac{39}{32} (h'(0))^2 \pi \Omega_3 dx'. \quad (3.111)$$

Case (13): $r = -2, \ell = -2, k = 0, j = 0, |\alpha| = 0$

From (2.20) and the Leibniz rule, we have

$$\begin{aligned} \text{Case (13)} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}\left[\pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \partial_{\xi_n} \sigma_{-2}(D^{-1})\right](x_0) d\xi_n \sigma(\xi') dx' \\ &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}\left[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \sigma_{-2}(D^{-1})\right](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \quad (3.112)$$

By **Case b** in [13] and Lemma 3.5, we obtain

$$\pi_{\xi_n}^+ \sigma_{-2}(D^{-1})(x_0)|_{|\xi'|=1} := B_1 - B_2, \quad (3.113)$$

where

$$\begin{aligned} B_1 &= h'(0) \frac{2+i\xi_n}{4(\xi_n-i)^2} c(\xi') c(dx_n) c(\xi') + h'(0) \frac{-i}{2(\xi_n-i)^2} c(\xi') + h'(0) \frac{-i\xi_n}{4(\xi_n-i)^2} c(dx_n) \\ &\quad + \frac{i}{4(\xi_n-i)^2} \partial_{x_n}[c(\xi')](x_0) + \frac{-(2+i\xi_n)}{4(\xi_n-i)^2} c(\xi') c(dx_n) \partial_{x_n}[c(\xi')](x_0), \end{aligned} \quad (3.114)$$

and

$$B_2 = \frac{h'(0)}{2} \left[\frac{c(dx_n)}{4i(\xi_n-i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n-i)^2} + \frac{3\xi_n - 7i}{8(\xi_n-i)^3} [ic(\xi') - c(dx_n)] \right]. \quad (3.115)$$

Hence in this case,

$$\begin{aligned} \partial_{\xi_n}(B_1) &= h'(0) \frac{-i\xi_n - 3}{4(\xi_n-i)^3} c(\xi') c(dx_n) c(\xi') + h'(0) \frac{i}{(\xi_n-i)^3} c(\xi') + h'(0) \frac{i\xi_n - 1}{4(\xi_n-i)^3} c(dx_n) \\ &\quad + \frac{-i}{2(\xi_n-i)^3} \partial_{x_n}[c(\xi')](x_0) + \frac{i\xi_n + 3}{4(\xi_n-i)^3} c(\xi') c(dx_n) \partial_{x_n}[c(\xi')](x_0), \end{aligned} \quad (3.116)$$

and

$$\partial_{\xi_n}(B_2) = h'(0) \frac{-2i\xi_n - 8}{8(\xi_n-i)^4} c(\xi') + h'(0) \frac{i\xi_n^2 + 4\xi_n - 9i}{8(\xi_n-i)^4} c(dx_n) \quad (3.117)$$

From Lemma 3.8, combining (3.77) and (3.116), we obtain

$$\text{trace}\left[\partial_{\xi_n}(B_1) \sigma_{-2}(D^{-1})\right](x_0) = \frac{2i\xi_n^5 - 10\xi_n^4 + 26i\xi_n^3 - 10\xi_n^2 + 37i\xi_n + 9}{-4(\xi_n-i)^3(1+\xi_n^2)^3} (h'(0))^2. \quad (3.118)$$

Combining (3.77) and (3.117), we obtain

$$\text{trace}\left[\partial_{\xi_n}(B_2) \sigma_{-2}(D^{-1})\right](x_0) = -\frac{2i\xi_n^6 + 8\xi_n^5 - 24i\xi_n^4 - 24\xi_n^3 - 35i\xi_n^2 - 68\xi_n + 27i}{4(\xi_n-i)^4(1+\xi_n^2)^3} (h'(0))^2. \quad (3.119)$$

Therefore

$$\begin{aligned}
\text{Case (13)} &= i(h'(0))^2 \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{16\xi_n^5 - 60i\xi_n^4 - 40\xi_n^3 - 82i\xi_n^2 - 114\xi_n + 36i}{4(\xi_n - i)^4(1 + \xi_n^2)^3} d\xi_n \sigma(\xi') dx' \\
&= i(h'(0))^2 \Omega_3 \int_{\Gamma^+} \frac{16\xi_n^5 - 60i\xi_n^4 - 40\xi_n^3 - 82i\xi_n^2 - 114\xi_n + 36i}{4(\xi_n - i)^4(1 + \xi_n^2)^3} d\xi_n dx' \\
&= i(h'(0))^2 \frac{2\pi i}{6!} \left[\frac{16\xi_n^5 - 60i\xi_n^4 - 40\xi_n^3 - 82i\xi_n^2 - 114\xi_n + 36i}{4(\xi_n + i)^3} \right]^{(6)} \Big|_{\xi_n=i} \Omega_3 dx' \\
&= -\frac{821}{256} (h'(0))^2 \pi \Omega_3 dx'.
\end{aligned} \tag{3.120}$$

Case (14): $r = -1$, $\ell = -3$, $k = 0$, $j = 0$, $|\alpha| = 0$

From (2.20) and the Leibniz rule, we have

$$\begin{aligned}
\text{Case (14)} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \partial_{\xi_n} \sigma_{-3}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
&= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \sigma_{-3}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'.
\end{aligned} \tag{3.121}$$

From (3.31), (3.82)-(3.87), (3.91) and direct computations, we obtain

$$\begin{aligned}
&\text{trace} \left[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \sigma_{-3}(D^{-1}) \right] (x_0) \Big|_{|\xi'|=1} \\
&= -\frac{-8\xi_n^7 + 18i\xi_n^6 - 12\xi_n^5 + 61i\xi_n^4 + 26\xi_n^3 + 66i\xi_n^2 + 78\xi_n - 25i}{2(\xi_n - i)^2(1 + \xi_n^2)^5} (h'(0))^2 \\
&\quad + h''(0) \frac{-2\xi_n^5 + 6i\xi_n^4 + 2\xi_n^3 + 11i\xi_n^2 + 14\xi_n - 5i}{(\xi_n - i)^2(1 + \xi_n^2)^4} \\
&\quad + s_{\partial_M} \frac{-\xi_n^3 - i\xi_n^2 + \xi_n - i}{2(\xi_n - i)^2(1 + \xi_n^2)^3} + \sum_{\alpha, l < n} R_{\alpha l \alpha}^{\partial_M}(x_0) \xi_i \xi_l \frac{i\xi_n^2 + 12\xi_n - 11i}{3(\xi_n - i)^2(1 + \xi_n^2)^3}.
\end{aligned} \tag{3.122}$$

Therefore

$$\begin{aligned}
\text{Case (14)} &= -i(h'(0))^2 \frac{2\pi i}{6!} \left[\frac{-8\xi_n^7 + 18i\xi_n^6 - 12\xi_n^5 + 61i\xi_n^4 + 26\xi_n^3 + 66i\xi_n^2 + 78\xi_n - 25i}{2(\xi_n + i)^5} \right]^{(6)} \Big|_{\xi_n=i} \Omega_3 dx' \\
&\quad + ih''(0) \frac{2\pi i}{5!} \left[\frac{-2\xi_n^5 + 6i\xi_n^4 + 2\xi_n^3 + 11i\xi_n^2 + 14\xi_n - 5i}{(\xi_n + i)^4} \right]^{(5)} \Big|_{\xi_n=i} \Omega_3 dx' \\
&\quad + s_{\partial_M} \frac{2\pi i}{4!} \left[\frac{-\xi_n^3 - i\xi_n^2 + \xi_n - i}{2(\xi_n + i)^3} \right]^{(4)} \Big|_{\xi_n=i} \Omega_3 dx' \\
&\quad + \sum_{i, l < n} R_{\alpha l \alpha}^{\partial_M}(x_0) \frac{\pi^2}{2} \delta_l^i \frac{2\pi i}{4!} \left[\frac{i\xi_n^2 + 12\xi_n - 11i}{3(\xi_n + i)^3} \right]^{(4)} \Big|_{\xi_n=i} dx' \\
&= \left(\frac{239}{64} (h'(0))^2 - \frac{27}{16} h''(0) + \frac{29}{192} s_{\partial_M} \right) \pi \Omega_3 dx'.
\end{aligned} \tag{3.123}$$

Case (15): $r = -3$, $\ell = -1$, $k = 0$, $j = 0$, $|\alpha| = 0$

From (2.20) we have

$$\text{Case (15)} = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\pi_{\xi_n}^+ \sigma_{-3}(D^{-1}) \partial_{\xi_n} \sigma_{-1}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \tag{3.124}$$

By the Leibniz rule, trace property and "++" and "--" vanishing after the integration over ξ_n in [9], then

$$\begin{aligned} & \int_{-\infty}^{+\infty} \text{trace} \left[\pi_{\xi_n}^+ \sigma_{-3}(D^{-1}) \partial_{\xi_n} \sigma_{-1}(D^{-1}) \right] d\xi_n \\ &= \int_{-\infty}^{+\infty} \text{trace} \left[\sigma_{-3}(D^{-1}) \partial_{\xi_n} \sigma_{-1}(D^{-1}) \right] d\xi_n - \int_{-\infty}^{+\infty} \text{trace} \left[\sigma_{-3}(D^{-1}) \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \right] d\xi_n. \end{aligned} \quad (3.125)$$

Combining these assertions, we see

$$\text{Case (15)} = \text{Case (14)} - i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\sigma_{-3}(D^{-1}) \partial_{\xi_n} \sigma_{-1}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.126)$$

By Lemma 3.2, a simple computation shows

$$\partial_{\xi_n} \sigma_{-1}(D^{-1})(x_0) \Big|_{|\xi'|=1} = \frac{-2\xi_n}{(1+\xi_n^2)^2} \sqrt{-1} c(\xi') + \frac{1-\xi_n^2}{(1+\xi_n^2)^2} \sqrt{-1} c(dx_n). \quad (3.127)$$

Combining (3.31), (3.82)-(3.87) and (3.127), we obtain

$$\begin{aligned} & \text{trace} \left[\sigma_{-3}(D^{-1}) \partial_{\xi_n} \sigma_{-1}(D^{-1}) \right] (x_0) \Big|_{|\xi'|=1} \\ &= \frac{8\xi_n^5 + 24\xi_n^3 + 28\xi_n^2}{(1+\xi_n^2)^5} (h'(0))^2 + \frac{-4\xi_n^3 - 8\xi_n}{(1+\xi_n^2)^4} h''(0) \\ &+ s_{\partial M} \frac{-\xi_n^5 + 4\xi_n^3 + \xi_n}{(1+\xi_n^2)^5} + \sum_{\alpha, i, l < n} R_{\alpha i l \alpha}^{\partial M}(x_0) \xi_i \xi_l \frac{20\xi_n}{3(1+\xi_n^2)^4}. \end{aligned} \quad (3.128)$$

We note that

$$\int_{-\infty}^{+\infty} \frac{8\xi_n^5 + 24\xi_n^3 + 28\xi_n^2}{(1+\xi_n^2)^5} d\xi_n = \frac{2\pi i}{4!} \left[\frac{8\xi_n^5 + 24\xi_n^3 + 28\xi_n^2}{(\xi_n + i)^5} \right]^{(4)} \Big|_{\xi_n=i} = 0, \quad (3.129)$$

and

$$\int_{-\infty}^{+\infty} \frac{-4\xi_n^3 - 8\xi_n}{(1+\xi_n^2)^4} d\xi_n = \int_{-\infty}^{+\infty} \frac{-\xi_n^5 + 4\xi_n^3 + \xi_n}{(1+\xi_n^2)^5} d\xi_n = \int_{-\infty}^{+\infty} \frac{20\xi_n}{3(1+\xi_n^2)^4} d\xi_n = 0. \quad (3.130)$$

Therefore

$$\text{Case (15)} = \left(\frac{239}{64} (h'(0))^2 - \frac{27}{16} h''(0) + \frac{29}{192} s_{\partial M} \right) \pi \Omega_3 dx'. \quad (3.131)$$

Now Φ is the sum of the **case** (1, 2, \dots , 15), so

$$\sum_{I=1}^{15} \text{case I} = \left(\frac{399}{256} (h'(0))^2 - \frac{29}{32} h''(0) + \left(\frac{71}{96} + \frac{3}{32} i \right) s_{\partial M} \right) \pi \Omega_3 dx'. \quad (3.132)$$

Hence we conclude that, for 5-dimensional compact manifold M with the boundary ∂M

$$\text{Vol}_5^{(1,1)} = \frac{1}{16} \int_{\partial M} \left(\frac{399}{16} (h'(0))^2 - \frac{29}{2} h''(0) + \left(\frac{71}{6} + \frac{3}{2} i \right) s_{\partial M} \right) \pi \Omega_3 d\text{vol}_{\partial M}. \quad (3.133)$$

Next we recall the Einstein-Hilbert action for manifolds with boundary (see [13] or [15]),

$$I_{\text{Gr}} = \frac{1}{16\pi} \int_M s d\text{vol}_M + 2 \int_{\partial M} K d\text{vol}_{\partial M} := I_{\text{Gr},i} + I_{\text{Gr},b}, \quad (3.134)$$

where

$$K = \sum_{1 \leq i, j \leq n-1} K_{i,j} g_{\partial M}^{i,j}; \quad K_{i,j} = -\Gamma_{i,j}^n, \quad (3.135)$$

and $K_{i,j}$ is the second fundamental form, or extrinsic curvature. Take the metric in Section 2, then by Lemma A.2 in [13], $K_{i,j}(x_0) = -\Gamma_{i,j}^n(x_0) = -\frac{1}{2}h'(0)$, when $i = j < n$, otherwise is zero. For $n = 5$, then

$$K(x_0) = \sum_{i,j} K_{i,j}(x_0) g_{\partial M}^{i,j}(x_0) = \sum_{i=1}^4 K_{i,i}(x_0) = -2h'(0). \quad (3.136)$$

So

$$I_{\text{Gr},b} = -4h'(0)\text{Vol}_{\partial M}. \quad (3.137)$$

On the other hand, by Proposition 2.10 in [27], we have

Lemma 3.9. *Let M be a 5-dimensional compact manifold with the boundary ∂M , then*

$$s_M(x_0) = 3(h'(0))^2 - 4h''(0) + s_{\partial M}(x_0). \quad (3.138)$$

Proof. From Proposition 2.10 in [27], let $B = [0, 1)$, $b^2 = \frac{1}{h(x_n)}$ and $F = \partial_M$, we obtain $s_B = 0$, $|\text{grad}_B b|^2 = (b')^2$ and

$$s_M(x_0) = 8b''(x_0) - 12(b'(x_0))^2 + s_{\partial M}(x_0). \quad (3.139)$$

By a simple computation, the lemma as follows. \square

Hence from (3.133), (3.136) and (3.138), we obtain

Theorem 3.10. *Let M be a 5-dimensional compact manifold with the boundary ∂M , then*

$$\widetilde{\text{Wres}}[(\pi^+ D^{-1})^2] = \frac{\pi^3}{16} \int_{\partial M} \left(\frac{225}{64} K^2 + \frac{29}{4} s_M|_{\partial M} + \left(\frac{197}{12} + 3i \right) s_{\partial M} \right) \text{dvol}_{\partial M}. \quad (3.140)$$

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References

- [1] V. W. Guillemin.: A new proof of Weyl's formula on the asymptotic distribution of eigenvalues. Adv. Math. 55, no. 2, 131-160, (1985).
- [2] M. Wodzicki.: local invariants of spectral asymmetry. Invent. Math. 75(1), 143-178, (1995).
- [3] M. Adler.: On a trace functional for formal pseudo-differential operators and the symplectic structure of Korteweg-de Vries type equations, Invent. Math. 50, 219-248, (1979).
- [4] A. Connes.: Quantized calculus and applications. XIth International Congress of Mathematical Physics (Paris, 1994), Internat Press, Cambridge, MA, 15-36, (1995).
- [5] A. Connes.: The action functional in Noncommutative geometry. Comm. Math. Phys. 117, 673-683, (1998). Rev. Math. Phys. 5, 477-532, (1993).
- [6] D. Kastler.: The Dirac Operator and Gravitation. Comm. Math. Phys. 166, 633-643, (1995).
- [7] W. Kalau and M. Walze.: Gravity, Noncommutative geometry and the Wodzicki residue. J. Geom. Physics. 16, 327-344, (1995).
- [8] T. Ackermann.: A note on the Wodzicki residue. J. Geom. Phys. 20, 404-406, (1996).
- [9] B. V. Fedosov, F. Golse, E. Leichtnam, E. Schrohe.: The noncommutative residue for manifolds with boundary. J. Funct. Anal. 142, 1-31, (1996).
- [10] E. Schrohe.: Noncommutative residue, Dixmier's trace, and heat trace expansions on manifolds with boundary. Contemp. Math. 242, 161-186, (1999).
- [11] Y. Wang.: Differential forms and the Wodzicki residue for Manifolds with Boundary. J. Geom. Physics. 56, 731-753, (2006).
- [12] S. W. Hawking.: In: S. W. Hawking and W. Israel. (eds.) General Relativity. Cambridge University Press, Cambridge, New York, (1979).
- [13] Y. Wang.: Gravity and the Noncommutative Residue for Manifolds with Boundary. Letters in Mathematical Physics. 80, 37-56, (2007).

- [14] R. Ponge.: Noncommutative Geometry and lower dimensional volumes in Riemannian geometry, *Lett. Math. Phys.* 83, 1-19 (2008).
- [15] Y. Wang.: Lower-Dimensional Volumes and Kastler-kalau-Walze Type Theorem for Manifolds with Boundary. *Commun. Theor. Phys.* Vol 54, 38-42, (2010).
- [16] J. Wang and Y. Wang.: Noncommutative residue and sub-Dirac operators for foliations, *J. Math. Phys.* 54, 012501 (2013).
- [17] J. Wang and Y. Wang.: The Kastler-Kalau-Walze type theorem for 6-dimensional manifolds with boundary, *arXiv: math.DG/1211.6223*.
- [18] T. Ackermann and J. Tolksdorf.: A generalized Lichnerowicz formula, the Wodzicki residue and gravity. *J. Geom. Physics.* 19, 143-150,(1996).
- [19] F. Pfäffle and C. A. Stephan.: On gravity, torsion and the spectral action principle. *J. Funct. Anal.* 262, 1529-1565,(2012).
- [20] F. Pfäffle and C. A. Stephan.: Chiral Asymmetry and the Spectral Action. *Commun. Math. Phys.* 321, 283-310 (2013).
- [21] J. Wang, Y. Wang, C. L. Yang.: Dirac operators with torsion and the noncommutative residue for manifolds with boundary. *J. Geom. Phys.* 81, 92-111 (2014).
- [22] Dobarro, F., Únal, B.: Curvature of multiply warped products. *J. Geom. Physics.* 55, 75-106 (2005).
- [23] N. Berline; E. Getzler; M. Vergne, Heat kernels and Dirac operators. Springer- Verlag, Berlin, 1992.
- [24] Y. Yu.: The Index Theorem and The Heat Equation Method, Nankai Tracts in Mathematics-Vol.2, World Scientific Publishing, (2001).
- [25] W. Zhang.: Local index theorem of Atiyah-Singer for families of Dirac operators. *Lect. Notes in Math.* 1369, 351-366. Springer, Verlag, (1989).
- [26] G. Grubb, E. Schrohe.: Trace expansions and the noncommutative residue for manifolds with boundary. *J. Reine Angew. Math.* 536, 167-207, (2001).
- [27] Y. Wang.: Curvature of multiply warped products with an affine connection. To appear in *Abstract and Applied Analysis* (2014).